Progressions

SEQUENCE
A sequence is a function whose domain is the set N of natural numbers.
REAL SEQUENCE
A Sequence whose range is a subset of R is called a real sequence.
In other words, a real sequence is a function with domain N and the range a subset of the set R of real numbers.

PROGRESSIONS: It is not necessary that the terms of a sequence always follow a Certain pattern or they are described by some explicit formula for the nth term. Those sequences whose terms follow certain patterns are called progressions.

ILLUSTRATION 1 **11, 7, 3, -1, …is an A.P. whose first term is 11 and the common difference 7-11=-4.**

ILLUSTRATION 2 Sow that the sequence $<a_n>$ defined by $a_n = 2n^2 +1$ is not an A.P.

PROPERTIES OF AN ARITHMETIC

PROPERTY I: If a is the first term and d the common difference of an A.P., then its nth terms a_n is given by $a_n = a+(n-1)d$

PROPERTY II: A sequence is an A.P iff its nth term is of the form An+B i.e. a linear expression in n. The common difference in such a case is A i.e. the coefficient of n.

PROPERTY III: If a constant is added to or subtracted from each term of an A.P., then the resulting sequence is also A.P. with the same common difference.

PROPERTY IV: If each term of a given A.P. is multiplied or divided by a non-zero constant k, then the resulting sequence is also an A.P. with common difference kd or d/k, where d is the common difference of the given A.P.

PROPERTY V: In a finite A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term i.e.

$a_k + a_{n-(k-1)} = a_1 + a_n$ for all k=1, 2, 3, …, n-1.

PROPERTY VI: Three numbers a, b, c are in A.P. iff 2b=a+c.

PROPERTY VII: If the terms of an A.P. are chosen at regulr intervals then they from an A.P.

PROPERTY VIII: If a_n, a_{n+1} and a_{n+2} are three consecutive terms of an A.P., then $2a_{n+1} = a_n+a_{n+2}$.

INSERTION OF ARITHMETIC MEANS
If between two given quantities a and b we have to insert n quantities $A_1, A_2, \ldots, A_n$ such that a, $A_1, A_2, \ldots, A_n$, b form an A.P., then we say that $A_1, A_2, \ldots, A_n$ are arithmetic means between a and b.

INSERTION OF n ARITHMETIC MEANS BETWEEN a AND b.
Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between two quantities a and b. Then,
a, $A_1, A_2, \ldots, A_n$, b is an A.P.
Let d be the common difference of this A.P. Clearly, it contains (n+2) terms.
$\therefore$ b = (n+2)th term

$\Rightarrow$ b = a+(n+1) d

$\Rightarrow \quad d = \dfrac{b-a}{n+1}$

Now, A_1 = a + d

$\Rightarrow A_1 = A_1 = \left(a + \dfrac{b-a}{n+1}\right)$

$A_2 = a + 2d \Rightarrow A_2 = \left(a + \dfrac{2(b-a)}{n+1}\right)$

$A_n = a + nd \Rightarrow A_n = A_n = \left(a + \dfrac{n(b-a)}{n+1}\right)$

ILLUSTRATIVE EXAMPLES

(1) Between 1 and 31 are inserted m arithmetic means so that the ratio of the 7^{th} and (m-1)th means is 5 : 9. Find the value of m.

GEOMETRIC PROGRESSION

A sequence of non-zero numbers is called a geometric progression (abbreviate as G.P.) if the ratio of a term and the term preceding to it is always a constant quantity.

The constant ratio is called the common ratio of the G.P.

In other words a sequence, $a_1, a_2, a_3, \ldots a_n, \ldots$ is called a geometric progression if

$\dfrac{a_n+1}{a_n}$ = constant for all $n \in$ N.

PROPERTIES OF GEOMETIC PROGRESSIONS

In this section, we shall discuss some properties of geometric progressions and geometric series

PROPERTY I: If all the terms of a G.P. be multiplied or divided by the same non-zero constant, then it remains G.P. with the same common ratio.

PROPERTY II: The reciprocals of the terms of a given G.P. form a G.P.

PROPERTY III: If each terms of a G.P. be raised to the same power, the resulting sequence also forms a G.P.

PROPERTY IV: In a finite G.P the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.

PROPERTY V Three non-zero numbers a, b, c are in G.P. iff $b^2 = ac$

PROPERTY VI If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.

PROPERTY VII If $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a G.P. of nonzero non negative terms, then $\log a_1, \log a_2, \ldots, \log a_n \ldots$ is an A.P and vice – versa.

ILLUSTRATIVE EXAMPLES

(1) Find all the complex numbers x and y such that x, x+2y, 2x+y are in A.P. and $(y+1)^2$, xy + 5, $(x+1)^2$ are in G.P. Also, find the progression.

(2) If mth term of a G.P.is m and nth term is n, then prove that its rth term is $\left(\frac{m^{r-n}}{n^{r-m}}\right)^{1/m-n}$

SUM OF n TERMS OF A G.P.

THEOREM **To prove that the sum of n terms of a G.P. With first term 'a' and common ratio 'r' is given by $S_n = a\left(\frac{r^n-1}{r-1}\right)$ or, $S_n = a\left(\frac{1-r^n}{1-r}\right), r \neq 1$.**

ILLUSTRATIVE EXAMPLES

(1) Let S denote the sum of the terms of an infinite G.P. and σ^2 denote the sum of the squares of the terms. Show that the sum of the first n terms of this geometric progression is given by $S\left[1-\left(\frac{S^2-\sigma^2}{S^2+\sigma^2}\right)^n\right]$

(2) Find the geometric progression of real number such that the sum of its first four terms is equal to 30 and the sum of the squares of the first four terms is 340.

(3) Prove that $\underbrace{111.....1}_{91\,times}$ is not a prime number.

INSERTION OF n GEOMETRIC MEANS BETWEEN TWO GIVEN NUMBERS A AND b.

Let G1, G2, … Gn be n geometric means between two given numbers a and b. Then,

A, G_1, G_2, … .,G_n, b is a G.P. consisting of (n+2) terms. Let r be the common ratio of this G.P. Then,

B = (n+2)th term = ar^{n+1}

$\Rightarrow \quad r^{n+1} = \frac{b}{a}$

$\Rightarrow \quad r=(b/a)^{1/n+1}$

$\therefore \quad G_1 = ar = a\left(\frac{b}{a}\right)^{1/(n+1)}$,

$\therefore \quad G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/(n+1)}$ ...,

$G_1 = ar^n = a\left(\frac{b}{a}\right)^{(n+1)}$

AN IMPORTANT PROPERTY OF GEOMETRIC GEANS THEOREM:

If geometric means are inserted between two quantities, then the product of n geometric means is the nth power of the single geometric mean between the two quantities.

SOME IMPORTANT PROPERTIES OF ARITHMETIC AND GEOMETRIC MEANS BETWEEN TWO GIVEN QUANTITIES.

PROPERTY I If A and G are respectively arithmetic and geometric means between two positive number a and b, then A > G.

PROPERTY II If A and G are respectively arithmetic and geometric means between two positive quantities a and b, then the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$

PROPERTY III If A and G be the A.M. and G.M. between two positive numbers, then the numbers are $A \pm \sqrt{A^2 - G^2}$

ILLUSTRATIVE EXAMPLES

(1) If one geometric mean G and two arithmetic means A_1 and A_2 be inserted between two given quantities, prove that $G^2 = (2A_1 - A_2)(2A_2 - A_1)$.

(2) The A.M. between m and n and the G.M. between a and b are each equal to $\frac{ma+nb}{m+n}$. Find m and n in terms of a and b.

ARITHMETICO-GEOMETRIC SERIES Let a, (a + d)r, $(a+2d)r^2$, $(a+3d)r^3$, … be an arithmetico – geometric sequence. Then, $a+(a+d)r+(a+2d)r^2+(a+3d)r^3+\ldots$ is an arithmetico geometric series.

ILLUSTRATION Find the nth term of the series $1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\ldots$

SUM OF n TERMS OF AN ARITHMETICO-GEOMETRIC SEQUENCE

THEOREM The sum of n terms of an arthmetico-geometric sequence
a, (a + d) r, $(a+2d)r^2$, $(a+3d)r^3$, … is given by

$$S_n = \begin{cases} \frac{a}{1-r} + dr\frac{(1-r^{n-1})}{1-r} - \frac{\{a+(n-1)d\}r^n}{1-r}, \; when\ r \neq 1 \\ \frac{n}{2}[2a+(n-1)d], \; when\ \ r = 1 \end{cases}$$

EXAMPLE Show that the sum of the series $1+\left(\frac{2n+1}{2n-1}\right)+5\left(\frac{2n+1}{2n-1}\right)^2+\ldots to$ n terms is an even or an odd number according as n is even or odd.

HARMONIC PROGRESSION
DEFINITION A sequence $a_1, a_2, \ldots, a_n, \ldots$ Of non-zero number is called a Harmonic progression, if the sequence $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}, \ldots$ is an Arithmetic progression .

nth TERM OF A HP
The nth term of a H.P is the reciprocal of the nth term of the corresponding A.P. Thus, if $a_1, a_2, a_3, \ldots, a_n \ldots$ is a HP and the common difference of the corresponding AP is d
i.e. $d = \frac{1}{a_{n+1}} - \frac{1}{a_n}$, then $a_n = \frac{1}{\frac{1}{a_1}+(n-1)d}$

If a, b, c are in HP, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP. Therefore $\frac{2}{b} = \frac{1}{a}+\frac{1}{c} \Rightarrow b = \frac{2ac}{a+c}$.

(1) If S_1, S_2 and S_3 denote the sum up to n(>1) terms of three non-constant sequence in A.P., whose first terms are unity and common differences are in H.P., prove that $n = \frac{2S_1S_3 - S_1S_2 - S_2S_3}{S_1 - 2S_2 + S_3}$

(2) If a, b, c are in HP and a > c, show that $\frac{1}{b-c}+\frac{1}{a-b} > \frac{4}{a-c}$

(3) Let a, b, c be positive real numbers. If a, A_1, A_2, b are in A.P., a, G_1, G_2, b are in G.P. and a, H_1, H_2, b are in H.P., show that $\frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2} = \frac{(2a+b)(a+2b)}{9ab}$

PROPERTIES OF ARITHMETIC, GEOMETRIC AND HARMONIC MEANS BETWEEN TWO GIVEN NUMBERS
Let A, G and H be arithmetic, geometric and harmonic means of two positive numbers a and b. Then.
$A = \frac{a+b}{2}, G\sqrt{ab}$ and $H = \frac{2ab}{a+b}$. These three means possess the following properties :

PROPERTY1 $H = \frac{2ab}{a+b}$ A $\geq$ G $\geq$ H.

PROPERTY2 A, G, H form a GP i.e, G^2 – AH.

PROPERTY3 The equation having a and b as its roots x^2 – 2Ax + G^2 = 0

PROPERTY4 If A, G, H are arithmetic, geometric and harmonic means between three given numbers a, b and c, then the equation having a, b, c as its roots is

$$x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

(1) For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the harmonic mean of a and b?

(2) If A_1, A_2; G_1, G_2; H_1, H_2 be two A.M.'s, G.M's and H.M.s between two number a and b, then prove that :

(3) If H_1, H_2,….,H_n be n harmonic means between a and b and n is a root of the equation x^2(1-ab)-x(a^2+b^2)-(1+ab)=0, then prove that H_1 = ab(a-b) $\frac{H_1}{H_n} = \frac{n+r}{nr+1}$

METHOD OF DIFFERENCES: Sometimes the nth term of a sequence or a series cannot be determined by the methods discussed in the earlier sections. In such cases, we use the following steps to find the nth term T_n of the given sequence.

STEP 1 Obtain the terms of the sequence and compute the differences between the successive terms of the given sequence. If these differences are in A.P, then take T_n = an^2 + bc +c, where a, b, c are constants. Determine a, b, c by putting n=1, 2, 3 and putting the values of T_1, T_2, T_3.

STEP 2 If the successive differences computed in step 1 are in G.P. with common ratio r, then take T_n = ar^{n-1} + bn+c.

STEP 3 If the differences of the differences computed in step 1 are in A.P., then take T_n = an^3 + bn^2 + cn + d and find the values of constants a, b, c, d.

STEP 4 If the differences of the differences computed in step 1 are in G.P. with common ratio r, then take T_n = ar^{n-1} + bn^2 + cn + d The following examples will illustrate the above procedure.

(1) Sum the following series to n terms : 5 + 7+ 13 + 31 + 85 +….

(2) If $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ and,

$$H_n' = \frac{n+1}{2} - \left[\frac{1}{n(n-1)} + \frac{2}{(n-1)(n-2)} + \frac{3}{(n-2)(n-3} + ... + \frac{n-2}{2.3}\right]$$

show that H_n = H_n^1

(3) Find the sum of first n terms of the series whose nth term is $\frac{1}{n(n+1)}(n^4 + 2n^3 + 2n^2 + n - 1)$

(4) If $\sum_{r=1}^{n} T_r = \frac{n(n+1)(n+2)(n+3)}{12}$, where T_r denotes the rth term of the series. Find, $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{T_r}$.

(5) Sum the series to n terms: $\frac{1}{(1+x)(1+2x)} + \frac{1}{(1+2x)(1+3x)} + \frac{1}{(1+3x)(1+4x)} + ...$