Tangents and Normals

ILLUSTRATIVE EXAMPLES:

(i) The curve $y=ax^3+bx^2+cx+5$ touches the x-axis at P(-2, 0) and cuts the y-axis at the point Q where its gradient is 3. Find the equation of the curve completely.

(ii) Find the equation of the normal to the curve $y= (1=x)^y + \sin^{-1}(\sin^2 x)$atx=0

(iii) Determine the constant c such that the straight line joining the points (0, 3) and 95, -2) is tangent to the curve $y=\frac{c}{x+1}$

(iv) Prove that all normal to the curve x=a cost + at sint, y = a sint - at cost

(v) Find the points at which the tangents to the curves $y=x^3 - x - 1$ and $y=3x^2 - 4x + 1$ are parallel. Also, find the equations of tangents.

(vi) Find the equation of the tangent to $x^3 = ay^2$ at the point A (at^2, at^3). Find also the point where this tangent meets the curve again.

(vii) Tangent at point P_1 (other than (0, 0) on the curve $y=x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3 and so on. Show that the abscissae of P_1, P_2, P_3. …. .. P_n form a GP. Also, find the ratio $\frac{area\ \Delta P_1P_2P_3}{area\ \Delta P_2P_3P_4}$

(viii) For the function F(x) = $\int_0^x 2|t|$ dt, find the tangent lines which are parallel to the bisector of the angle in the first quadrant.

(ix) If α, β are the intercepts made on the axes by the tangent at any point of the curve $x=a\cos^3\theta$, $y=b\sin^3\theta$, prove that $\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1$.

(x) If x_1 and y_1 be the intercepts on the axes of X an Y cut off by the tangent to the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=1,$ then prove that $\left(\frac{a}{x_1}\right)^{n/n-1}+\left(\frac{b}{y_1}\right)^{n/n-1}=1.$

(xi) Show that the normal to the rectangular hyperbola $xy=c^2$ at the point P $\left(ct_1, \frac{c}{t_1}\right)$ meets the curve again at the point $Q\left(ct_2, \frac{c}{t_2}\right)$, if $t_1^3\, t_2 = -1$

ANGLE OF INTERSECTION OF TWO CURVES

The angle of intersection of two curves is defined to be the angle between the tangents to the two curves at their point of intersection.

ORTHOGONAL CURVES

If the angle of intersection of two curves is a right angle, the two curves are said to intersect orthogonally and the curves are called orthogonal curves.

If the curves C_1 and C_2 are orthogonal, then $\phi = \pi/2$

$\therefore$ $m_1 m_2 = -1 \Rightarrow \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2}=-1$

EXAMPLES:

(i) Find the acute angle between the curves $y = |x^2 - 1|$ and $y=|x^2 - 3|$ at their points of intersection

(ii) Show that the curves $x^3 - 3xy^2 = -2$ and $3x^2y-y^3=2$ cut orthogonally.

(iii) Find the acute angles between the curves $y = |2x^2 - 4|$ and $y=|x^2 - 5|$.

(iv) Show that the curves $y^2=4ax$ and $ay^2=4x^3$ intersect each other at an angle of $\tan^{-1}\frac{1}{2}$ and also if PG_1 and PG_2 be the normals to two curves at common point of intersection (other than the origin) meeting the axis of X in G_1 and G_2, then
$G_1 G_2 = 4a$.

LENGTHS OF TANGENT, NORMAL, SUBTANGENT AND SUBNORMAL

Let the tangent and normal at a point P(x, y) on the curve y=f(x), meet the x-axis at T and N respectively. If G is the foot of the ordinate at P, then TG and GN are called the Cartesian subtangent and subnormal, while the lengths PT and PN are called the lengths of the tangent and normal respectively.

If PT makes angle Ψ with x-axis, then $\tan\Psi = \frac{dy}{dx}$. From Fig we find that

Subtangent = TG = y cot $\Psi = \dfrac{y}{\left(\dfrac{dy}{dx}\right)}$

Subnormal = GN = y tan $\Psi = y\dfrac{dy}{dx}$

Length of the tangent = PT = y cosec Ψ

$$= y\sqrt{1+\cot^2\Psi}$$