Probability

ELEMENTARY EVENT
If a random experiment is performed, then each of its outcomes is known as an elementary event.

SAMPLE SPACE The set all possible outcomes of a random experiment is called the sample space associated with it and it is generally denoted by S.

ILLUSTRATION Consider the experiment of tossing two coins together or a coin twice.

In this experiment the possible outcomes are.
Head on first and Head on second
Head on first and Tail on second,
Tail on first and Head on second,
Tail on first and Tail on second.

If we define
HH = Getting head on both coins,
HT = Getting head on first and tail on second
TH = Getting tail on first and head on second,
TT = Getting tail on both coins.

COMPOUND EVENT A subset of the sample space associated to a random experiment is said to define a compound event if it is disjoint union of single element subsets of the sample space.

NEGATION OF AN EVENT Corresponding to every event A associated with a random experiment we define an event "not A" which occurs when and only when A does not occur.

PROBABILITY
DEFINITION If there are n elementary events associated with a random experiment and m of them are favourable to an event A, then the probability of happing or occurrence of A is denoted by P(A) and is defined as ratio $\frac{m}{n}$.

Thus, P(A) = $\frac{m}{n}$

Clearly, $0 \le m \le n$. Therefore,

$$0 \le \frac{m}{n} \le 1$$

$\Rightarrow \quad 0 \le p(a) \le 1$

If p(A) =1, then A is called certain event and A is called an impossible event, if P(A) = 0.

The number of elementary events which will ensure the non-occurrence A i.e. Which ensure the occurrence of $\bar{A}$ is (n-m). Therefore.

$$P(\bar{A}) = \frac{n-m}{n}$$

$$= 1 - \frac{m}{n}$$

$$= 1 - P(A)$$

$\Rightarrow \quad$ P(A) + P$(\bar{A})$ = 1

The odds in favour of occurrence of the event A are defined by m : (n-m) i.e, P(A) : P$(\bar{A})$ and the odds against the occurrence of A are defined n-m : m i.e, P$(\bar{A})$: P(A)

1. An unbiased die, with face numbered 1, 2, 3, 4, 5, 6, is thrown n times and the list of n number showing up is noted. What is the probability that, among numbers, 1, 2, 3, 4, 5, 6, only three numbers appear in this list?
2. Three six faced die are thrown together. Find the probability that the sum of the numbers appearing on them is $k(9 \le k \le 14)$.
3. In a bag there are three tickets numbered 1, 2, 3. A ticket is drawn at random and put back, and this is done four times. Find the probability of getting an even number as the sum.

GEOMETRICAL PROBABILITY

1. Two points are taken on a straight line AB of length unity. Prove that the probability that the distance between them exceeds ($0<l<1$) is $(1-l)^2$.
2. The outcomes of an experiment are represented by points in the square bounded by x=0, y=0, x=2 and y=2 in the xy plane. If the probability be distributed uniformly, determine the probability that $x^2 + y^2 > 1$.

ALGEBRA OF EVENTS

In this section, we shall see how new events can be constructed by combining two or More events associated to a random experiment.

Let A and B be two events associated to a random experiment with sample space S. We Define the event "A or B" which is said to occur iff an elementary event favourable to either A or B or both is an outcome. In other words, the event "A r B" occurs iff either A or B or both occur i.e. at least one of A and B occurs. Thus, "A or B" is represented by the subset $P((AP((A\cap\bar{B})\cup(\bar{A}\cap B))\bar{B})\cup(\bar{A}\cap B))$ of the sample space S.

Give verbal descriptions of some events and their equivalent set theoretic notations for ready reference.

Verbal description of the event	**Equivalent set theoretic notation**
Not A	$\bar{A}$
A or B (at least one of A or B)	$A\cup B$
A and B	$A \cap B$
A but not B	$A\cap\bar{B}$
Neither A nor B	$\bar{A}\cap\bar{B}$
At least one of A, B or C	$A\cup B\cup C$
Exactly one of A and B	$(A\cap\bar{B})\cup(\bar{A}\cap B)$
All three of A, B and C	$A\cap B\cap C$
Exactly two of A, B and C	$(A\cap B\cap\bar{C})\cup(A\cap\bar{B}\cap C)\cup(\bar{A}\cap B\cap C)$

ILLUSTRATION Let A, B and C are three arbitrary events. Find the expression for the events noted below, in the context of A, B and C.

(i) Only A occurs
(ii) Both A and B, but not C occur
(iii) All the three events
(iv) At least one occurs
(v) At least two occur
(vi) One and no more occurs
(vii) Two and no more occurs I
(viii) None occurs
(ix) Not more than two occur.

SOLUTION (i) $A\cap\bar{B}\cap C$
(ii) $A\cap B\cap\bar{C}$
(iii) $A\cap B\cap C$ I
(iv) $A\cup B\cup C$
(v) $(A\cap\bar{B})\cup(B\cap C)\cup(A\cap C)$
(vi) $(A\cap\bar{B}\cap\bar{C})\cup(\bar{A}\cap B\cap\bar{C})\cup(\bar{A}\cap\bar{B}\cap C)$
(vii) $(A\cap B\cap\bar{C})\cup(\bar{A}\cap B\cap C)\cup(A\cap\bar{B}\cap C)$
(viii) $(\bar{A}\cap\bar{B}\cap\bar{C} = \overline{A\cup B\cup C}$

(ix) $(A\cap B)\cup(B\cap C)\cup(A\cap C)-3(A\cap B\cap C)$

TYPE OF EVENTS

All events associated to a random experiment are divided into different types on the basis of their nature of occurrence. In this section, we shall discuss those types.

MUTUALLY EXCLUSIVE EVENTS: Two or more events associated to a random experiment are mutually exclusive if the occurrence of one of them prevents or denies the occurrence all others. .

It follows from the above definition that two or more events associated to a random experiment are mutually exclusive, if there is no elementary event which is favrourable to all the events.

Thus, if two events A and B are mutually exclusive, then $P(A\cap B)=0$

Similarly, if A, B and C are mutually exclusive events, then $P(A\cap B\cap C)=0$.

EXHAUSTIVE EVENTS Two or more events associated to a random experiment are exhaustive if their union is the sample space. i.e. events A_1, A_2,…., A_n associated to a random experiment with sample space S are exhaustive if $A_1\cup A_2...\cup A_n=S$.

INDEPENDENT EVENTS Two event A and B associated to a random experiment are independent if the probability of occurrence or non occurrence of A is not affected by the occurrence or non-occurrence of B.

THEOREM 1 (Addition Theorem for two events) If A and B are two events associated with a random experiment, then $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

COROLLARY If A and B are mutually exclusive events, then there fore $P(A\cap B)=0,\ P(A\cup B)=P(A)+P(B)$ This is the addition theorem for mutually exclusive events.

THEOREM 2 (addition Theorem for three events). If A, B, C are three events associated with a random experiment, then $P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$

COROLLARY If A, B, C are mutually exclusive events, then $P(A\cap B)=P(B\cap C)=P(A\cap C)$ = $P(A\cap B\cap C)=0$.

$\therefore\quad P(A\cup B\cup C)=P(A)+P(B)+P(C)$

THEOREM 3 Let A and B be two events associated to a random experiment. Then,

(i) $P(\bar{A}\cap B)=P(B)-P(A\cap B)$ (ii) $P(A\cap\bar{B})=P(A)-P(A\cap B)$

(iii) $P((A\cap\bar{B})\cup(\bar{A}\cap B))$ = P(A)+P(B)-2P(A∩B)

THEOREM 4 For any two events A and B, prove that

$$P(A\cap B)\le P(A)\le P(A\cup B)\le P(A)+p(B).$$

THEOREM 4 For any two events A and B, Prove that the probability that exactly one of A, B occurs is given by

$$P(A)+P(B)-2P(A\cup B)=P(A\cup B)-P(A\cap B)$$

THEOREM 5 If A, B, C are three events, then prove that

(i) P(At least two of A, B, C occur)

= $P(A\cap B)+P(B\cap C)+P(C\cap A)-2P(A\cap B\cap C)$

(ii) P (exactly two of A, B, C occur)

= $P(A\cap B)+P(B\cap C)+P(A\cap C)-3P(A\cap B\cap C)$

(iii) P (Exactly one of A, B, C occurs)

$P(A)+P(B)+P(C)-2P(A\cap B)-2P(B\cap C)-2P(A\cap C)+3P(A\cap B\cap C)$

Example. A die is loaded so that the probability of face I is proportional to i, i =1,2,… .6. What is the probability of an even number occurring when the die is rolled?

CONDITIONAL PROBABILITY

Let A and B be two events associated with a random experiment. Then, the probability of occurrence of event A under the condition that B has already occurred and P(B) $\neq 0$, is called the conditional probability and it is denoted by P(A/B). Thus, we have
P(A/B)= Probability of occurrence of A given that B has already occurred.
Similarly, P(B/A) when P(A) $\neq 0$ is defined as the probability of occurrence of event B when A has already occurred.

ILLUSTRATION 1: Let there be a bag containing 5 white and 4 red balls. Two balls are drawn from the bag one after the other without replacement. Consider the following events
A = Drawing a white ball in the first draw,
B = Drawing a red ball in the second draw.
Now,
P(B/A) = Probability of drawing a red ball in second draw given that a white ball has already been drawn in the first draw
= Probability of drawing a red ball from a bag containing 4 white and 4 red balls.

ILLUSTRATION 2: Two integers are selected at random from integers 1 to 11. If the sum is even, find the probability that both the numbers are odd.

MULTIPLICATION THEOREMS ON PROBABILITY

In this section, we shall discuss some theorems which are helpful in computing the probabilities of simultaneous occurrences of two or more events associated with a random experiment.

THEOREM 1 If A and B are two events associated with a random experiment, then
$P(A\cap B)=P(A)P(B/A)$, if P(A) $\neq$ 0
$P(A\cap B)=P(B)P(A/B)$, if P(B) $\neq 0$.

THEOREM 2 (Extension of multiplication theorem) if $A_1, A_2,\ldots,A_n$ are n events associated with a random experiment, then

$P(A_1\cap A_2\cap A_3....\cap A_n)=P(A_1)P(A_2/A_1)P(A_3/A_1\cap A_2)$

...... $P(A_n/A_1\cap A_2\cap....\cap A_{n-1})$, where $P(A_i/A_1\cap A_2....\cap A_{i-1})$ represents the conditional probability of the occurrence of event A_i, given that the events $A_1, A_2 \ldots A_{i-1}$ have already occurred.

1. Suppose n persons are asked a question successively in a random order and exactly 3 of the n persons know the answer.
   (i) If n > 6, find the probability that the first four of those asked do not know the answer.
   (ii) Show that the probability that the rth person asked is the fist to know the answer is $\frac{3(n-r)(n-r-1)}{n(n-1)(n-2)}$ where $1\le r\le n-2$
2. A die loaded so that the probability of throwing the number is proportional to i. Find the probability that the number 2 has occurred, given that when the die is recalled an even number has turned up.

MORE ON INDEPENDENT EVENTS
In section 40.6, we have defined independent events and we have seen that two events and B are independent if P(B/A)=P(B) and P(A/B)=P(A).

Also, $P(A\cap B)=P(A)P(B)$ if A and B are independent events.
In this section, we shall discuss about pair wise independence and mutual independence

of events.

PAIRWISE INDEPENDENT EVENTS: Let $A_1, A_2, \ldots, A_n$ be n events associated to a random experiment. These events are said to be pair wise independent if

$P(A_1 \cap A_j) = P(A_i)P(A_j)$ for $i \neq j;\ i, j = 1, 2..., n$

$P(A_i \cap A_j \cap A_k) = P(A_i)P(A_j)P(A_k),$ for $i \neq j \neq k;$ j, k = 1,2,….,n

$\vdots$

$P(A_1 \cap A_2 ..... \cap A_n) = P(A_1)P(A_2)....P(A_n)$

ILLUSTRATION 1 A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B, C are defined as
A : "the first bulb is defective",
B : "the second bulb is non-defective",
C : "the second bulb is non-defective",
Determine whether (i) A, B, C are pair wise independent,
(ii) A, B, C are mutually independent.

THEOREM If A and B are independent events associated with a random experiment, then prove that
(i) $\overline{A}$ and B are independent events (ii) A and $\overline{B}$ are independent events
(iii) $\overline{A}$ and $\overline{B}$ are also independent events

1. For three independent events A, B and C, the probability to A to occur is a, the probability that A, B and C will not occur is b, and the probability that at last one of thee three events will not occur is c. If p denotes the probability that c occurs but neither A nor B occurs, prove that p is a root of the equation

   ap^2 + {ab +(1-a)(1-a-c)} p+b(1-a)(1-c)=0 and deduces that $c > \dfrac{(1-a)^2 + ab}{1-a}$
2. An urn contains five balls alike in every respect except colour. If three of these balls are white and two are black and we draw two balls at random from this urn without replacing them. If A is the event that the first ball drawn is white and B the event that the second ball drawn is black, are A and B independent?

THE LAW OF TOTAL PROBABILITY

THEOREM (Law of total probability) Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ….or E_n, then

$$P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \ldots P(E_n)\, P(A/E_n)$$
$$= \sum_{r=1}^{n} P(E_r)\ P(A/E_r).$$

1. Urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. What is the probability that the ball drawn is now white?
2. An employer sends a letter to his employee but he does not receive the reply (It is certain that the employee would have replied if he did receive the letter). It is known that one out n letters does not reach its destination. Find the probability that the employee does not receive the letter.

BAYE'S RULE

THEOREM (Baye's Theorem) Let S be the sample space and let $E_1, E_2 \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or….or E_n, then $P(E_k / A) = \dfrac{P(A/E_k).P(E_k)}{\sum_{i=1}^{n} P(A/E_i).P(E_i)}$

1. A company has two plants to manufacture scooters. Plant 1 manufactures 70% of the scooters and Plant II manufactures 30%. At Plant I, 80% of the scooters are rated as of standard quality and at Plant II, 90% of the scooters are rated as of standard quality A scooter is chosen at random and is found to be of standard quality. What is the probability that it has come from Plant II?

RANDOM VARIABLE AND ITS PROBABILITY DISTRIBUTION

DEFINITION Let S be the sample space associated with a given random experiment. Then a real valued function X which assigns to each even $w \in S$ to a unique real number X (w) is called a random variable. .

In other words, a random variable is a real valued function having domain as the sample space associated with a random experiment.

MEAN AND VARIANCE OF A RANDOM VARIABLE

MAEN If X is discrete random variable which assumes values $x_1, x_2, x_3, \ldots, x_n$ with respective probabilities $p_1, p_2, p_3, \ldots, p_n$ then the mean $\overline{X}$ of X is defined as

$$\overline{X} = p_1 x_1 + p_2 x_2 + \ldots + p_n x_n \text{ or, } \overline{X} = \sum_{i=1}^{n} p_i x_i$$

VARIANCE If X is a discrete random variable which assumes values $x_1, x_2, x_3, \ldots, x_n$ With the respective probabilities $p_1, p_2, \ldots, p_n$, then variance of X is defined as

Var (X) = $p_1(x_1 - \overline{X})^2 + p_2(x_2 - \overline{X})^2 + \ldots + p_n(x_n - \overline{X})^2 \ldots \ldots + p_n(x_n - \overline{X})^2$

$= \sum_{i=1}^{n} p_i (x_i - \overline{X})^2$, Where $\overline{X} = \sum_{i=1}^{n} p_i x_i$ is the mean of X.

Now,

$$= \sum_{i=1}^{n} p_i x_i^2 - \overline{X}^2$$

BINOMIAL DISTRIBUTION A random variable X which takes values 0. 1, 2,.., n is said to follow binomial distribution if its probability distribution function is given by

P(X = r) = ${}^nC_r p^r q^{n-r}$, $r = 0, 1, 2, \ldots n$, where p, q > 0 such that p + q = 1

1. An urn contains 25 balls of which 10 balls bear a mark ‘A’ and the remaining 15 balls bear a mark ‘B’. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that
   (i) all will bear ‘ A’ marked
   (ii) not more than 2 will bear ‘B’ mark
   (iii) then number of balls with ‘A’ mark and ‘B ’ mark will be equal
   (iv) at least one ball will bear ‘B’ mark
2. Numbers are selected at random one at a time, from the numbers 00, 01, 02,…, 99 with replacement. An event A occurs if the product of the two digits of the selected number is 18. If four numbers are selected, find the probability that A occurs at lest 3 time.