CIRCLES

1. The set of points in a plane which are at a constant distance 'r' (≥ 0) from a given point C is called a ***circle***. The fixed point C is called the ***centre*** and the constant distance r is called the ***radius*** of the circle.
2. A circle is said to be a ***unit circle*** if its radius is 1 unit.
3. A circle is said to be a ***point circle*** if its radius is zero. A point circle contains only one point, the centre of the circle.
4. The equation of the circle with centre C (a, b) and radius 'r' is $(x - a)^2 + (y - b)^2 = r^2$.
5. The equation of a circle simplest form is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$. The equation of a circle with centre origin and radius 'r' is $x^2 + y^2 = r^2$.
6. If $g^2 + f^2 - c \geq 0$ then the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.
7. The conditions that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a circle are (i) $a = b$; (ii) $h = 0$; (iii) $g^2 + f^2 - ac \geq 0$.
8. If $ax^2 + ay^2 + 2gx + 2fy + c = 0$ represents a circle, then its centre = $(-g/a, -f/a)$ and its radius = $\sqrt{g^2 + f^2 - ac} / |a|$.
9. We use the following notation in circles.
   $S \equiv x^2 + y^2 + 2gx + 2fy + c$
   $S_1 \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
   $S(x_1, y_1) = S_{11} \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
   $S_{12} \equiv x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c$
10. Let S = 0 be a circle and $P(x_1, y_1)$ be a point. Then
    i) P lies inside the circle $S = 0 \Leftrightarrow S_{11} < 0$
    ii) P lies on the circle $S = 0 \Leftrightarrow S_{11} = 0$
    iii) P lies outside the circle $S = 0 \Leftrightarrow S_{11} > 0$
11. The power of a point P(x, y) with respect to the circle S = 0 is S_{11}.
12. Let S = 0 be a circle with centre C and radius 'r'. Let P be a point. Then $CP^2 - r^2$ is called power of P with respect to the circle S = 0.
13. Let S = 0 be a circle and P be a point. Then
    i) P lies inside the circle $S = 0 \Rightarrow S_{11} < 0$
    ii) P lies in the circle $S = 0 \Rightarrow s_{11} = 0$
    iii) P lies outside the circle $S = 0 \Rightarrow S_{11} > 0$
14. The equation of a circle having the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.
15. Two circles are said to be ***concentric*** if their centres are the same.
16. The equation of a circle concentric with the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is of the form $x^2 + y^2 + 2gx + 2fy + k = 0$, where k is a constant.
17. Given 3 points A, B, and C then
    i) only one circle passes through A, B, and C iff A, B, C are non collinear.
    ii) A circle through A, B, C is impossible iff A, B, C are collinear
18. The equation of the circumcircle of the triangle formed by the line $ax + by + c = 0$ with the coordinate axes is $ab(x^2 + y^2) + c(bx + ay) = 0$.

The general form of equation of the circle circumscribing the triangle formed by the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ is $a(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) + b(a_2x + b_2y + c_2)(a_3x + b_3y + c_3) + c(a_3x + b_3y + c_3)(a_1x + b_1y + c_1) = 0$.

If two lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ meet the coordinate axes in four distinct points then those points are concyclic $\Leftrightarrow a_1a_2 = b_1b_2$.

19. If the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ meet the coordinate axes in four distinct concyclic points, then the equation of the circle passing through these concyclic points is $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) - (a_1b_2 + a_2b_1)xy = 0$.
20. The equation of the chord joining the two points $A(x_1, y_1)$, $B(x_2, y_2)$ in the circle $S = 0$ is $S_1+S_2 = S_{12}$.
21. The equation of the tangent to the circle $S = 0$ at $P(x_1, y_1)$ is $S_1 = 0$.
22. The equation of the normal to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at $P(x_1, y_1)$ is $(y_1 + f)(x - x_1) - (x_1 + g)(y - y_1) = 0$.
23. The normal to the circle $S = 0$ at $P(x_1, y_1)$ passes through the centre $(-g, -f)$ of the circle.
24. The equation of the normal to the circle $x^2 + y^2 = r^2$ at $P(x_1, y_1)$ is $y_1x - x_1y = 0$.
25. Let $L = 0$ be a straight line and $S = 0$ be a circle with centre C and radius 'r'. Let d be the perpendicular distance from C to the line $L = 0$. Then

    i) $L = 0$ touches the circle $S = 0 \Leftrightarrow r = d$.

    ii) $L = 0$ intersects the circle $S = 0 \Leftrightarrow r > d$. Let
    $L = 0$ be a line and $S = 0$ be a circle with centre C and radius 'r'. Let d be the perpendicular distance from C to the line $L = 0$. If $r > d$ then
    $L=0$ is a chord of the circle $S = 0$. The length of the chord $= 2\sqrt{r^2 - d^2}$. If $r < d$ then $L = 0$ do not intersect the circle $S = 0$.

    iii) $L = 0$ does not touch or intersect the circle
    $S = 0 \Leftrightarrow r < d$.
26. The condition for the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ to touch the coordinate axes is $g^2 = f^2 = c$.
27. The condition for the straight line $y = mx + c$ to touch the circle $x^2 + y^2 = r^2$ is $c^2 = r^2(1 + m^2)$.
28. The condition for the x-axis to touch the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ $(c > 0)$ is $g^2 = c$.
29. The condition for the y-axis to touch the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ $(c > 0)$ is $f^2 = c$.
30. The condition for the straight line $lx + my + n = 0$ may be a tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(g^2 + f^2 - c)(l^2 + m^2) = (lg + mg - n)^2$.
31. If the straight line $y = mx + c$ touches the circle $x^2 + y^2 = r^2$, then their point of contact is $\left(-\frac{r^2m}{c}, \frac{r^2}{c}\right)$.
32. The equation of a tangent to the circle $x^2 + y^2 = r^2$ may be taken as $y = mx \pm r\sqrt{1+m^2}$.
    The condition that the straight line $lx + my + n = 0$ may touch the circle $x^2 + y^2 = r^2$ is $n^2 = r^2(l^2 + m^2)$ and the point of contact is $\left(\frac{-r^2l}{n}, \frac{-r^2m}{n}\right)$.
33. Let $S = 0$ be a circle with centre (a, b) and radius 'r'. Then

    i) $S = 0$ touches x-axis $\Leftrightarrow r = |b|$

    ii) $S = 0$ touches y-axis $\Leftrightarrow r = |a|$

    iii) $S = 0$ touches both the axes $\Leftrightarrow r = |\alpha| = |\beta|$
34. If the tangent drawn from an external point P to a circle $S = 0$ touches the circle at A then PA is called ***length of tangent*** from P to the circle $S=0$.

35. The length of the tangent drawn from an external point $P(x_1, y_1)$ to the circle S =0 is $\sqrt{S_{11}}$.
36. The length of the intercept made by the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ on (i) x-axis is $2\sqrt{g^2 - c}$ (ii) y-axis is $2\sqrt{f^2 - c}$.
37. If a line passing through a point $P(x_1, y_1)$ intersects the circle S = 0 at the points A and B then $PA \cdot PB = S_{11}$.
38. If A, B, C, D are four points of which no three are collinear such that $PA \cdot PC = PB \cdot PD$ for some point P then the point D lies on the circle passing through A, B, C (ie., A, B, C, D are concyclic).
39. Two tangents can be drawn to a circle from an external point.
40. The line joining the points of contact of the tangents to a circle S = 0 drawn from an external point P is called ***chord of contact*** of P with respect to S = 0.
41. The equation to the chord of contact of $P(x_1, y_1)$ with respect to the circle S = 0 is $S_1 = 0$.
42. The locus of the point of intersection of the tangents to the circle S = 0 drawn at the extremities of the chord passing through a point P is a straight line L = 0, called the ***polar*** of P with respect to the circle S = 0. The point P is called the ***pole*** of the line L = 0 with respect to the circle S=0.
43. The equation of the polar of the point $P(x_1, y_1)$ with respect to the circle S = 0 is $S_1 = 0$.
44. If P lies outside the circle S = 0 then the polar of P meets the circle in two points and the polar becomes the chord of contact of P.
45. If P lies on the circle S = 0 then the polar of P becomes the tangent at P to the circle S = 0.
46. If P lies inside the circle S = 0, then the polar of P does not meet the circle in any point.
47. If P is the centre of the circle S = 0, then the polar of P with respect to S = 0 does not exist.
48. The pole of the line $lx + my + n = 0$ $(n \neq 0)$ with respect to $x^2 + y^2 = r^2$ is $\left(\frac{-r^2 l}{n}, \frac{-r^2 m}{n}\right)$.
49. Two points P and Q are said to be ***conjugate points*** with respect to the circle S = 0 if the polar of P with respect to S = 0 passes through Q.
50. The condition for the points $P(x_1, y_1)$, $Q(x_2, y_2)$ to be conjugate with respect to the circle S = 0 is $S_{12} = 0$.
51. Two lines $L_1 = 0$, $L_2 = 0$ are said to be ***conjugate*** with respect to the circle S = 0 if the pole of $L_1 = 0$ lies on $L_2 = 0$.
52. The condition for the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ to be conjugate with respect to the circle $x^2 + y^2 = r^2$ $(l_1 l_2 + m_1m_2) = n_1n_2$.
53. Let S = 0 be a circle with centre C and radius r. Two points P, Q are said to be ***inverse points*** with respect to S = 0 if i) C,P, Q are collinear (ii) P, Q lies on the same side of C (iii) $CP \cdot CQ = r^2$.
54. If P, Q are a pair of inverse points with respect to a circle S = 0 then Q is called ***inverse point*** of P.
55. Let S = 0 be a circle with centre C and radius 'r'. The polar of a point P with respect to the circle S = 0 meets $\overleftrightarrow{CP}$ in Q iff P, Q are inverse points with respect to S = 0.
56. If P, Q are inverse points with respect to S = 0 then P, Q are conjugate points with respect to S = 0.
57. If P, Q are inverse points with respect to S = 0 then Q is the foot of the perpendicular from P on the polar of P with respect S = 0.
58. The polar of a point P with respect to a circle with centre C is a perpendicular to $\overleftrightarrow{CP}$.
59. The equation of the chord of the circle S = 0 having as its midpoint is $S_1 = S_{11}$.
60. The equation to the pair of tangents to the circle S = 0 from $P(x_1, y_1)$ is $S_1^2 = S_{11}S$.
61. If P(x, y) is a point on the circle with centre $C(\alpha, \beta)$ and radius r, then $x = \alpha + r\cos\theta$, $y = \beta + r\sin\theta$ where $0 \leq \theta < 2\pi$.

62. The equations $x = \alpha + r\cos\theta$, $y = \beta + r\sin\theta$, $0 \le \theta < 2\pi$ are called ***parametric equations*** of the circle with centre (α, β) and radius r.
63. A point on the circle $x^2 + y^2 = r^2$ is taken in the form $(r \cos \theta, r \sin \theta)$. The point $(r \cos \theta, r \sin \theta)$ is simply denoted as point θ.
64. The equation of the chord joining two points θ_1 and θ_2 on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(x + g) \cos \frac{\theta_1+\theta_2}{2} + (y+f)\sin\frac{\theta_1+\theta_2}{2} = r\cos\frac{\theta_1-\theta_2}{2}$, where r is radius of circle.
65. The equation of the tangent at $P(\theta)$ on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(x + g)\cos\theta + (y + f)\sin\theta = \sqrt{g^2+f^2-c}$.
66. The equation of the tangent at $P(\theta)$ on the circle $x^2 + y^2 = r^2$ is $x \cos \theta + y \sin \theta = r$.
67. The equation of the normal at $P(\theta)$ on the circle $x^2 + y^2 = r^2$ is $x \sin \theta - y \cos \theta = 0$.
68. If (x_1, y_1) is one end of a diameter of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then the other end is $(-2g - x_1, -2f - y_1)$.
69. The area of the triangle formed by the tangent at (x_1, y_1) on the circle $x^2 + y^2 = a^2$ with the coordinate axes is $\frac{a^4}{2|x_1y_1|}$.
70. If $lx + my = 1$ touches the circle $x^2 + y^2 = a^2$ then $l^2 + m^2 = a^{-2}$.
71. The pole of the line $lx + my + n = 0$ with respect to the circle $(x-\alpha)^2 + (y-\beta)^2 = r^2$ is $\left(\alpha - \frac{r^2 l}{N}, \beta - \frac{r^2 m}{N}\right)$ where $N = l\alpha + m\beta + n$.
72. If A and B are conjugate points with respect to a circle S = 0 and l_1, l_2 are the lengths of tangents from A, B to S = 0, then $AB^2 = l_1^2 + l_2^2$.
73. The middle point of the chord intercepted on the line $lx + my + n = 0$ by the circle $x^2 + y^2 = a^2$ is $\left(\frac{-ln}{l^2+m^2}, \frac{-mn}{l^2+m^2}\right)$.
74. The length of the intercept cut of from the line $ax + by + c = 0$ by the circle $x^2 + y^2 = r^2$ is $2\sqrt{\left[\frac{r^2(a^2+b^2)-c^2}{a^2+b^2}\right]}$.
75. If (x_1, y_1) is the midpoint of the chord AB of the circle S = 0 then length of AB is $2\sqrt{-S_{11}}$.
76. If (x_1, y_1) is the midpoint of the chord AB of the circle S = 0 and the tangents at A, B meet at C then the area of ΔABC is $\frac{(-S_{11})^{3/2}}{\sqrt{S_{11}+r^2}}$ where r is the radius of the circle.
77. The locus of midpoint of the chord of a circle S = 0, parallel to L = 0 is the diameter of S = 0 and which is perpendicular to L = 0.
78. If θ is the angle between the pair of tangents drawn from (x_1, y_1) to the circle S = 0 of radius r then $\tan\frac{\theta}{2} = \frac{r}{\sqrt{S_{11}}}$.

79. If $l_1x + m_1y + n_1 = 0$, $l_2x + m_2y + n_2 = 0$ are conjugate lines w.r.t tue circle $x^2 + y^2 + 2gx + 2fy + c = 0$ then, $(l_1 l_2 + m_1m_2)(g^2 + f^2 - c) = (g l_1 + fm_1 - n_1)$ $(g l_2 + fm_2 - n_2)$
80. The length and the midpoint of the chord $lx + my + n = 0$ $(n \neq 0)$ w.r.t the circle $x^2 + y^2 = a^2$ is $2\sqrt{\frac{a^2(\ell^2 + m^2) - n^2}{\ell^2 + m^2}}$, $\left(\frac{-\ell n}{\ell^2 + m^2}, \frac{-mn}{\ell^2 + m^2}\right)$.
81. The condition that the pair of tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ may be at right angles is $g^2 + f^2 = 2c$.
82. EQ of the circle passing through (a, b), (a, a) and (b, a) is $x^2 + y^2 - x(a + b) - y(a + b) + 2ab = 0$.
83. If two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ meet the coordinate axes in four distinct points then those points are concylic if $a_1a_2 = b_1b_2$ and its centre is $\left(\frac{\text{sum of } x - \text{intercepts}}{2}, \frac{\text{sum of } y - \text{intercepts}}{2}\right)$.
84. A square is inscribed in the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with its sides parallel to the axes of coordinates. The coordinates of the vertices are $\left(-g \pm \frac{r}{\sqrt{2}}, -f \pm \frac{r}{\sqrt{2}}\right)$ and its side $a = \sqrt{2}\,r$.
85. An equilateral triangle is inscribed in the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ then

    i) the area of circle = $\frac{3\sqrt{3}}{4}(g^2 + f^2 - c)$

    ii) side $a = \sqrt{3}\,r$
86. The farthest distance of an external point $p(x_1.y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is cp + r.
87. The farthest point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ form an external point $P(x_1, y_1)$ is B which divides centre c and p in the ratio r : cp + r externally.
88. The nearest point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from an external point $p(x_1, y_1)$ is A which divides centre c and p in the ratio r : cp – r internally.
89. The locus of the point of intersection of two perpendicular tangents to $s \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is $s - r^2 = 0$.
90. Area of the triangle formed by tangent at (x_1, y_1) to s = 0 with coordinate axes is $\frac{1}{2}\frac{|gx_1 + fy_1 + c|^2}{|x_1 + g||y_1 + f|}$.
91. Tangents from a point are drawn one to each concentric circle $s_1 = 0$ and $s_2 = 0$. If the tangents are perpendicular then the locus of the points is $(x + g)^2 + (y + f)^2 = r_1^2 + r_2^2$.
92. For any point on the circle $x^2 + y^2 = a^2$ tangents are drawn to the circle $x^2 + y^2 = b^2$ $(a > b)$ then the angle between the tangents is $2\sin^{-1}(b/a)$.
93. The area of the Quadrilateral formed by the two tangents through $P(x_1, y_1)$ to the circle and centre is $r\sqrt{s_{11}}$.
94. The angle subtended by the midpoint of chord at the centre of the circle is $\theta = 2\cos^{-1}(d/r)$.
95. The locus of the mid points of chords of the circle s = 0 makes an angle 90° at the centre of the circle is $(x + g)^2 + (y + f)^2 = r^2/2$