16. PARTIAL DIFFERENTIATION

Synopsis :

1. The partial derivative of f(x,y) with respect to x is the derivative of f(x, y) with respect to x by treating y as constant. Also the partial derivative of f(x, y) with respect to y is the derivative of f(x, y) with respect to y by treating x as constant.
2. If v = g(u), u = f(x, y) then

   (i) $\frac{\partial v}{\partial x} = \frac{dv}{du} \cdot \frac{\partial u}{\partial x}$. (ii) $\frac{\partial v}{\partial y} = \frac{dv}{du} \cdot \frac{\partial u}{\partial y}$.
3. If u = f(x, y) then $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ is called **total differential** of u = f(x, y). It is denoted by du or df.
4. If u = f(x, y), x = $g_1(t)$, y = $g_2(t)$ then $\frac{\partial f}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$ is called **total differential coefficient** of u = f(x, y) with respect to t.
5. If f(x, y) = c, where c is a constant then $\frac{df}{dx} = -\frac{\partial f}{\partial x} / \frac{\partial f}{\partial y}$.
6. If the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ of u = f(x, y) exist then $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are also functions of two variables. If the partial derivatives of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ exists then they are called **second order partial derivatives** of u = f(x, y) and these are denoted by

   (i) $\frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial x}\right\} = \frac{\partial^2 u}{\partial x^2} = f_{x,x} = f_{xx}$

   (ii) $\frac{\partial}{\partial y}\left\{\frac{\partial u}{\partial x}\right\} = \frac{\partial^2 u}{\partial y \partial x} = f_{y,x} = f_{yx}$

   (iii) $\frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial y}\right\} = \frac{\partial^2 u}{\partial x \partial y} = f_{x,y} = f_{xy}$

   (iv) $\frac{\partial}{\partial y}\left\{\frac{\partial u}{\partial y}\right\} = \frac{\partial^2 u}{\partial y^2} = f_{y,y} = f_{yy}$
7. If u = f(x, y) is continuous on its domain then $f_{xy} = f_{yx}$.
8. If f(x, y) = c where c is a constant then $\frac{d^2y}{dx^2} = -\left(\frac{f_{xy}f_y^2 - 2f_{xy}f_xf_y + 2f_{xy}f_x^2}{f_y^3}\right)$
9. A function u = f(x, y) is said to be a **homogeneous function** of **degree** or **order** n in x, y if f(kx, ky) = k^nf(x, y).

10. If u = f(x, y) is a homogeneous function of degree n then $u = x^n g\left(\frac{y}{x}\right)$, where g is a function of y/x.

11. If u = f(x, y) is a homogeneous function of degree n then $u = y^n h\left(\frac{x}{y}\right)$, where h is a function of x/y.

12. Euler's Theorem: If u = f(x, y) is homogeneous function of degree n in x, y then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

13. If u = f(x, y, z) is a homogeneous function of degree n then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$.

14. If u = f(x_1, x_2, ..., x_r) is a homogeneous function of degree n then $x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} + ...x_r\frac{\partial u}{\partial x_r} = nu.$

15. If u = f(x, y) is a homogeneous function of degree n then

i) $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = (n-1)\frac{\partial u}{\partial x}$

ii) $x\frac{\partial^2 u}{\partial y\,\partial x} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$

iii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$