Calculus:

Introduction:

Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my "class notes" they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I've tried to make these notes as self contained as possibleand so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes fromsemester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don't always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in classthat leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Here is a listing and brief description of the material in this set of notes.

- Review
- Limits
- Derivatives
- Applications of Derivatives
- Integrals
- Applications of Integrals
- Extras
- Vector Functions

Review:

Technically a student coming into a Calculus class is supposedto know both Algebra and Trigonometry. The reality is often much different however. Most students enter a Calculus class woefully unprepared for both the algebra and the trig that is in a Calculus class. This is very unfortunate since good algebraskills are absolutely vital to successfully completing anyCalculus course and if your Calculus course includes trig (as this one does) good trig skills are also important in many sections.

The intent of this chapter is to do a very cursory review of some algebra and trig skills that are absolutely vital to acalculus course. This chapter is not inclusive in the algebra and trig skills that are needed to be successful in a Calculus course. It only includes those topics that most students are particularly deficient in. For instance factoring is also vital to completing a standard calculus class but is not included here. For a more in depth review you should visit my Algebra/Trig review or my full set of Algebra notes at http://tutorial.math.lamar.edu.

Note that even though these topics are very important to aCalculus class I rarely cover all of these in the actual class itself. We simply don't have the time to do that. I do cover certain portions of this chapter in class, but for the most part I leave it to the students to read this chapter on their own.

Here is a list of topics that are in this chapter. I've also denoted the sections that I typically cover during the first couple of days of a Calculus class.

Review : Functions - Here is a quick review of functions, function notation and a couple of fairly important ideas about functions.

Review : Inverse Functions - A quick review of inverse functions and the notation for inverse functions.

Review : Trig Functions - A review of trig functions, evaluation of trig functions and the unit circle. This section usually gets a quick review in my class.

Review : Solving Trig Equations - A reminder on how to solve trig equations. This section is always covered in my class.

Review : Solving Trig Equations with Calculators, Part I- The previous section worked problem whose answers were always the "standard" angles. In this section we work some problems whose answers are not "standard" and so acalculator is needed. This section is always covered in my class as most trig equations in the remainder will need a calculator.

Review : Solving Trig Equations with Calculators, Part II- Even more trig equations requiring a calculator to solve.

Limits:

The topic that we will be examining in this chapter is that of Limits. This is the first of three major topics that we will becovering in this course. While we will be spending the least amount of time on limits in comparison to the other two topics limits are very important in the study of Calculus. We will be seeing limits in a variety of places once we move out of this chapter. In particular we will see that limits are part of the formal definition of the other two major topics.

Here is a quick listing of the material that will be covered in this chapter.

Tangent Lines and Rates of Change - In this section we willtake a look at two problems that we will see time and again in this course. These problems will be used to introduce the topic of limits.

The Limit - Here we will take a conceptual look at limits and try to get a grasp on just what they are and what they can tell us.

One-Sided Limits - A brief introduction to one-sided limits.

Limit Properties - Properties of limits that we'll need to use in computing limits. We will also compute some basic limits in this section

Computing Limits - Many of the limits we'll be asked to compute will not be "simple" limits. In other words, we won't be able to just apply the properties and be done. In this section we will look at several types of limits that require some work before we can use the limit properties to compute them.

Infinite Limits - Here we will take a look at limits that have a value of infinity or negative infinity. We'll also take a brief look at vertical asymptotes.

Derivatives:

In this chapter we will start looking at the next major topic in acalculus class. We will be looking at derivatives in this chapter (as well as the next chapter). This chapter is devoted almost exclusively to finding derivatives. We will be looking at one application of them in this chapter. We will be leaving most ofthe applications of derivatives to the next chapter.

Here is a listing of the topics covered in this chapter.

The Definition of the Derivative - In this section we will be looking at the definition of the derivative.

Interpretation of the Derivative - Here we will take a quick look at some interpretations of the derivative.

Differentiation Formulas - Here we will start introducing some of the differentiation formulas used in a calculus course.

Product and Quotient Rule - In this section we will took at differentiating products and quotients of functions.

Derivatives of Trig Functions - We'll give the derivatives of the trig functions in this section.

Derivatives of Exponential and Logarithm Functions - In this section we will get the derivatives of the exponential and logarithm functions

Applications of Derivatives:

In the previous chapter we focused almost exclusively on the computation of derivatives. In this chapter will focus onapplications of derivatives. It is important to always remember that we didn't spend a whole chapter talking about computingderivatives just to be talking about them. There are many very important applications to derivatives.

The two main applications that we'll be looking at in this chapter are using derivatives to determine information about graphs of functions and optimization problems. These will notbe the only applications however. We will be revisiting limits and taking a look at an application of derivatives that will allow us to compute limits that we haven't been able to compute previously. We will also see how derivatives can be used to estimate solutions to equations.

Here is a listing of the topics in this section.

Rates of Change - The point of this section is to remind us of the application/interpretation of derivatives that we were dealing with in the previous chapter. Namely, rates of change.

Critical Points - In this section we will define critical points. Critical points will show up in many of the sections in this chapter so it will be important to understand them.

Minimum and Maximum Values - In this section we will take a look at some of the basic definitions and facts involving minimum and maximum values of functions.

Finding Absolute Extrema - Here is the first application ofderivatives that we'll look at in this chapter. We will be determining the largest and smallest value of a function on an interval.

The Shape of a Graph, Part I - We will start looking at the information that the first derivatives can tell us about the graph of a function. We will be looking at increasing/decreasing functions as well as the First Derivative Test.

The Shape of a Graph, Part II - In this section we will look at the information about the graph of a function that the secondderivatives can tell us. We will look at inflection points, concavity, and the Second Derivative Test.

Integrals:

In this chapter we will be looking at integrals. Integrals are the third and final major topic that will be covered in this class. As with derivatives this chapter will be devoted almost exclusively to finding and computing integrals. Applications will be given in the following chapter. There are really two types of integrals that we'll be looking at in this chapter : Indefinite Integrals and Definite Integrals. The first half of this chapter is devoted to indefinite integrals and the last half is devoted to definite integrals. As we will see in the last half of the chapter if we don't know indefinite integrals we will not be able to do definite integrals.

Here is a quick listing of the material that is in this chapter.

Indefinite Integrals - In this section we will start with the definition of indefinite integral. This section will be devoted mostly to the definition and properties of indefinite integrals.

Computing Indefinite Integrals - In this section we will compute some indefinite integrals and take a look at a quick application of indefinite integrals.

Substitution Rule for Indefinite Integrals - Here we will look at the Substitution Rule as it applies to indefinite integrals. Many of the integrals that we'll be doing later on in the course and in later courses will require use of the substitution rule.

More Substitution Rule - Even more substitution rule problems.

Area Problem - In this section we start off with the motivation for definite integrals and give one of the interpretations of definite integrals.

Definition of the Definite Integral - We will formally define the definite integral in this section and give many of its properties. We will also take a look at the first part of the Fundamental Theorem of Calculus.

Applications of Integrals:

In this last chapter of this course we will be taking a look at a couple of applications of integrals. There are many otherapplications, however many of them require integrationtechniques that are typically taught in Calculus II. We will therefore be focusing on applications that can be done only with knowledge taught in this course.

Because this chapter is focused on the applications of integralsit is assumed in all the examples that you are capable of doing the integrals. There will not be as much detail in the integration process in the examples in this chapter as there was in the examples in the previous chapter.

Here is a listing of applications covered in this chapter.

Average Function Value - We can use integrals to determine the average value of a function.

Area Between Two Curves - In this section we'll take a look at determining the area between two curves.

Volumes of Solids of Revolution / Method of Rings - This is the first of two sections devoted to find the volume of a solid of revolution. In this section we look that the method of rings/disks.

Volumes of Solids of Revolution / Method of Cylinders - This is the second section devoted to finding the volume of a solid of revolution. Here we will look at the method of cylinders.

More Volume Problems - In this section we'll take a look at find the volume of some solids that are either not solids of revolutions or are not easy to do as a solid of revolution.

Work - The final application we will look at is determining the amount of work required to move an object.

Extras:

In this chapter material that didn't fit into other sections for a variety of reasons. Also, in order to not obscure the mechanics of actually working problems, most of the proofs of various facts and formulas are in this chapter as opposed to being in the section with the fact/formula.

This chapter contains those topics.

Proof of Various Limit Properties : we prove several of the limit properties and facts that were given in various sections of the Limits chapter.

Proof of Various Derivative Facts/Formulas/Properties : this section we give the proof for several of the rules/formulas/properties of derivatives that we saw in Derivatives Chapter. Included are multiple proofs of the Power Rule, Product Rule, Quotient Rule and Chain Rule.

Proof of Trig Limits : we give proofs for the two limits that are needed to find the derivative of the sine and cosine functions.

Proofs of Derivative Applications Facts/Formulas : give proofs of many of the facts that we saw in the Applications of Derivatives chapter.

Proof of Various Integral Facts/Formulas/Properties :we will give the proofs of some of the facts and formulas from the Integral Chapter as well as a couple from the Applications of Integrals chapter.

Area and Volume Formulas : is the derivation of the formulas for finding area between two curves and finding the volume of a solid of revolution.

Vector Functions:

We first saw vector functions back when we were looking at theEquation of Lines. In that section we talked about them because we wrote down the equation of a line in $\mathbb{R}^3$ in terms of a **vector function** (sometimes called a **vector-valued function**). In this section we want to look a little closer at them and we also want to look at some vector functions in $\mathbb{R}^3$ other than lines.

A vector function is a function that takes one or more variablesand returns a vector. We'll spend most of this section looking at vector functions of a single variable as most of the places where vector functions show up here will be vector functions of single variables. We will however briefly look at vector functions of two variables at the end of this section.

A vector functions of a single variable in $\mathbb{R}^2$ and $\mathbb{R}^3$ have the form,

$$\vec{r}(t) = \langle f(t), g(t) \rangle \qquad\qquad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

respectively, where $f(t)$, $g(t)$ and $h(t)$ are called the**component functions**.

The main idea that we want to discuss in this section is that of graphing and identifying the graph given by a vector function. Before we do that however, we should talk briefly about the domain of a vector function. The **domain** of a vector function is the set of all *t*'s for which all the component functions are defined.

Example 1 Determine the domain of the following function.

$$\vec{r}(t) = \left\langle \cos t, \ln(4-t), \sqrt{t+1} \right\rangle$$

Solution

The first component is defined for all *t*'s. The secondcomponent is only defined for $t < 4$. The third component is only defined for $t \geq -1$. Putting all of these together gives the following domain.

$$[-1, 4)$$

This is the largest possible interval for which all three components are defined.

Let's now move into looking at the graph of vector functions. In order to graph a vector function all we do is think of the vector returned by the vector function as a position vector for points on the graph. Recall that a position vector, say $\vec{v} = \langle a, b, c \rangle$, is a vector that starts at the origin and ends at the point (a, b, c).

So, in order to sketch the graph of a vector function all we need to do is plug in some values of *t* and then plot points that correspond to the resulting position vector we get out of the vector function.

Because it is a little easier to visualize things we'll start off by looking at graphs of vector functions in $\mathbb{R}^2$.

Example 2 Sketch the graph of each of the following vector functions.

(a) $\vec{r}(t) = \langle t, 1 \rangle$

(b) $\vec{r}(t) = \langle t, t^3 - 10t + 7 \rangle$

Solution

(a) $\vec{r}(t) = \langle t, 1 \rangle$

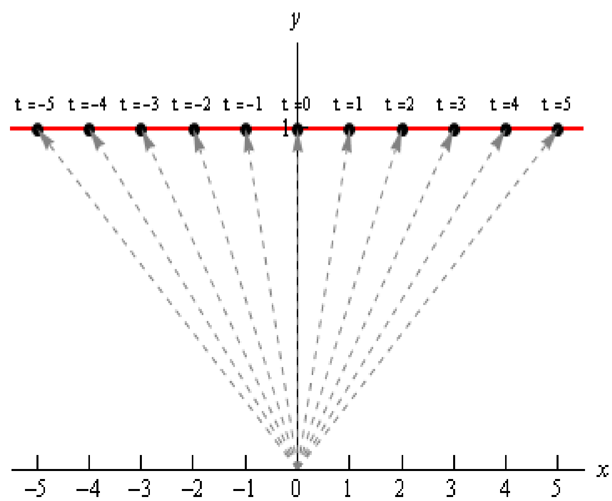
Okay, the first thing that we need to do is plug in a few values of *t* and get some position vectors. Here are a few,

$$\vec{r}(-3) = \langle -3, 1 \rangle \qquad \vec{r}(-1) = \langle -1, 1 \rangle \qquad \vec{r}(2) = \langle 2, 1 \rangle \qquad \vec{r}(5) = \langle 5, 1 \rangle$$

So, what this tells us is that the following points are all on the graph of this vector function.

$$(-3, 1) \qquad (-1, 1) \qquad (2, 1) \qquad (5, 1)$$

Here is a sketch of this vector function.



In this sketch we've included many more evaluations that just those above. Also note that we've put in the position vectors (in gray and dashed) so you can see how all this is working. Note however, that in practice the position vectors are generally not included in the sketch.

In this case it looks like we've got the graph of the line $y = 1$.

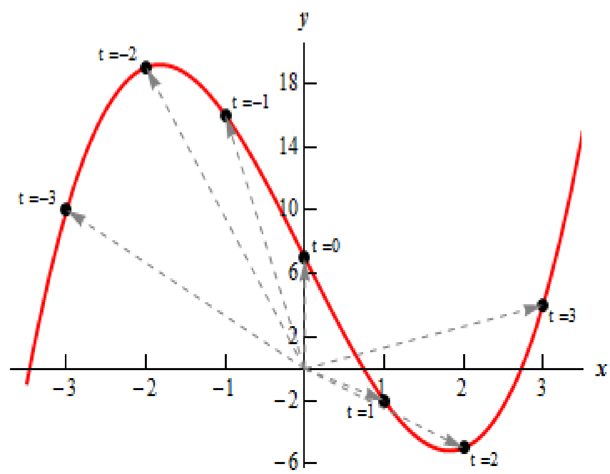
(b) $\vec{r}(t) = \langle t, t^3 - 10t + 7 \rangle$

Here are a couple of evaluations for this vector function.

$\vec{r}(-3) = \langle -3, 10 \rangle \qquad \vec{r}(-1) = \langle -1, 16 \rangle \qquad \vec{r}(1) = \langle 1, -2 \rangle \qquad \vec{r}(3) = \langle 3, 4 \rangle$

So, we've got a few points on the graph of this function. However, unlike the first part this isn't really going to be enough points to get a good idea of this graph. In general, it can take quite a few function evaluations to get an idea of what the graph is and it's usually easier to use a computer to do the graphing.

Here is a sketch of this graph. We've put in a few vectors/evaluations to illustrate them, but the reality is that we did have to use a computer to get a good sketch here.



y
x
t =−2
t =−1
t =−3
t =0
t =1
t =2
t =3
18
14
10
6
2
−2
−6
−3
−2
−1
1
2
3