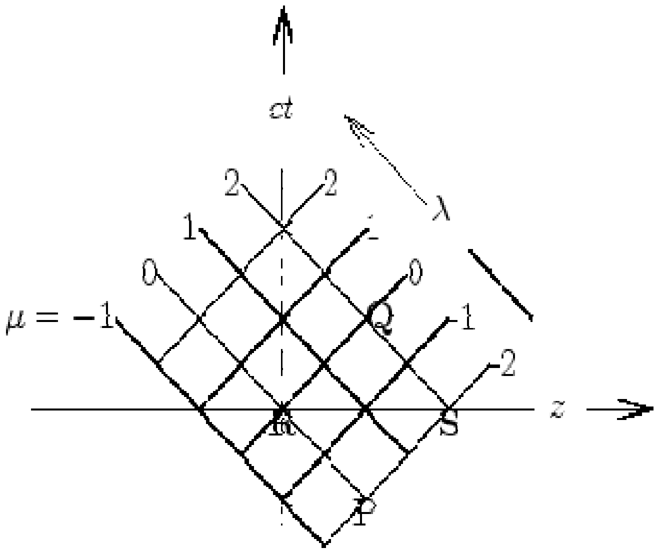


Figure 6.1: Characteristic coordinate lines μ and λ as determined by the wave equation for a simple string.

Note that f is constant along the λ characteristics (i.e., where $\lambda =$ constant), while g is constant along the μ characteristics. It follows that if f is known on the boundary segment RS , then f is known along all the λ -characteristics intersecting RS . Similarly, if g is known along RS , then g is known along all the μ -characteristics intersecting RS . And this is precisely the case because the Cauchy data on RS determine the values of both f and g on that segment.

Being the sum of the two functions, the solution to the waveequation is

$$\psi(z,t) = f(ct-z) + g(ct+z)$$

$$= \frac{1}{2}\psi_0(z-ct) + \frac{1}{2}\psi_0(z+ct) + \frac{1}{2c}\int_{z-ct}^{z+ct} V_0(z')\,dz' \qquad (616)$$

Thus one sees that any disturbance on a string consists of two parts: one propagating to the right the other to the left. The propagation speeds are $\pm c$, the slopes of the characteristicsrelative to the t - z coordinate system. The idiosyncratic aspect of the simple string is that these two parts do not change their shape as they propagate along the string.

A general linear hyperbolic system does not share this feature. However, what it does share with a simple string is that its solution is uniquely determined in the common region traversed by the two sets of characteristics which intersect RS . In fact, the Cauchy data on RS determine a unique solution $\psi(z,t)$ at every point in the region $PRQS$. This is why it is called the *domain of dependence*of RS . To justify these claims it is neccessary to construct this unique solution to a general second order linear hyperbolic differential equation.

System of Partial Differential Equations: How to Solve Maxwell's Equations Using Linear Algebra:

The theme of the ensuing development is linear algebra, but the subject is an overdetermined system of partial differential equations, namely, the Maxwell field equations. The objective is to solve them via the method of eigenvectors and eigenvalues. The benefit is that the task of solving the Maxwell system of p.d. equations is reduced to solving a single inhomogeneous scalar equation[64]

$$\left(\partial_x^2+\partial_y^2+\partial_z^2-\partial_t^2\right)=-4\pi S(t,\vec{x}) \ ,$$

where S is a time and space dependent source. The impatient reader will find that once this master equation, or its manifestation in another coordinate system, has been solved, the electric and magnetic fields are entirely determined as in Tables ❑-6.9.

The starting point of the development is Maxwell's equations. There is the set of four functions, the density of charge

$$\rho \qquad = \rho(\vec{x},t) \qquad \left[\frac{(\text{charge})}{(\text{volume})}\right] \tag{637}$$

and the charge flux

$$\vec{J} \quad = \vec{J}(\vec{x},t) \quad \left[\frac{(\text{charge})}{(\text{time})(\text{area})}\right], \tag{638}$$

which are usually given. These space and time dependent charge distributions give rise to electric and magnetic fields, $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$. The relationship is captured by means of Maxwell's gift to twentieth century science and technology,

$$\nabla \cdot \vec{B} \quad = 0 \quad (\text{"No magnetic monopoles"}) \tag{639}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \quad = 0 \quad (\text{"Faraday's law"}) \tag{640}$$

and

$$\nabla \cdot \vec{E} \quad = 4\pi\rho \quad (\text{"Gauss' law"}) \tag{641}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} \quad = 4\pi \vec{J} \quad (\text{"Ampere's law"}), \tag{642}$$

Maxwell's field equations[65].

Exercise 62.1 (Charge Flux-Density of an Isolated Charge)
Microscopic observations show that charged matter is composed of discrete point charges. On the other hand, macroscopic observations show that charged matter is the carrier of an electric fluid which is continuous. Dirac delta functions provide the means to grasp both attributes from a single perspective. This fact is highlighted by the following problem.

Consider the current-charge density due to an isolated moving charge,

$$\vec{J}(x, y, z, t) = q\int_{-\infty}^{\infty} \frac{d\vec{X}(\tau)}{d\tau}\delta(x - X(\tau))\delta(y - Y(\tau))\delta(z - Z(\tau))\delta(t - T(\tau))\, d\tau$$

$$\rho(x, y, z, t) = q\int_{-\infty}^{\infty} \frac{dT(\tau)}{d\tau}\delta(x - X(\tau))\delta(y - Y(\tau))\delta(z - Z(\tau))\delta(t - T(\tau))\, d\tau$$

a) Show that this current-charge density satisfies

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 .$$

Remark. The four-vector $\left(\frac{d\vec{X}(\tau)}{d\tau}, \frac{dT(\tau)}{d\tau}\right)$ is the charge's four-velocity in spacetime. The parameter τ is the ``wristwatch" time (as measured by a comoving clock) attached to this charge.

b) By taking advantage of the fact $\frac{dT(\tau)}{d\tau} > 0$, evaluate the τ-integrals, and obtain explicit expressions for the components $\vec{J}$ and ρ.

Answer:

$$\vec{J}(x, y, z, t) = \frac{d\vec{X}}{dt}\, q\,\delta(x - X(t))\delta(y - Y(t))\delta(z - Z(t)) = \dot{\vec{X}}(\tau) \text{ evaluated at } \tau \text{ as determined by } \delta(t - T(\tau)),$$

$$\rho(x, y, z, t) = q\,\delta(x - X(t))\delta(y - Y(t))\delta(z - Z(t))$$

where

Maxwell Wave Equation:

The first pair of Maxwell's equations, (6.39) and (6.40), imply that there exists a vector potential $\vec{A}$ and scalar potential ϕ from which one derives theelectric and magnetic fields,

$$\vec{B} = \nabla \times \vec{A} \tag{643}$$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} . \tag{644}$$

Conversely, the existence of these potentials guarantees that the first pair of these equations is satisfied automatically. By applying these potentials to the differential expressions of the second pair of Maxwell's equations, (6.41-6.42) one obtains the mapping

$$\begin{bmatrix} \phi \\ \vec{A} \end{bmatrix} \overset{\mathcal{A}}{\rightsquigarrow} \mathcal{A} \begin{bmatrix} \phi \\ \vec{A} \end{bmatrix} , \tag{645}$$

where

$$\mathcal{A}\begin{bmatrix} \phi \\ \vec{A} \end{bmatrix} \equiv \begin{bmatrix} -\nabla^2\phi - \nabla \cdot \dfrac{\partial \vec{A}}{\partial t} \\ \nabla \times \nabla \times \vec{A} + \nabla \dfrac{\partial \phi}{\partial t} + \dfrac{\partial^2 \vec{A}}{\partial t^2} \end{bmatrix} . \tag{646}$$

It follows that Maxwell's field equations reduce to Maxwell's four-component wave equation,

$$\begin{bmatrix} -\nabla^2\phi - \nabla\cdot\dfrac{\partial\vec{A}}{\partial t} \\ \nabla\times\nabla\times\vec{A} + \nabla\dfrac{\partial\phi}{\partial t} + \dfrac{\partial^2\vec{A}}{\partial t^2} \end{bmatrix} = 4\pi \begin{bmatrix} \rho \\ \vec{J} \end{bmatrix} . \tag{647}$$

Maxwell's wave operator is the linch pin of his theory of electromagnetism. This is because it has the following properties:

1. It is a *linear* map from the space of four-vector fields into itself, i.e.

$$R^4 \xrightarrow{\mathcal{A}} R^4$$

at each point event $(t, \vec{x})$.

2. The map is singular. This means that there exist nonzero vectors $\vec{\mathcal{U}}_r$ and $\vec{\mathcal{U}}_\ell$ such that

$$\mathcal{A}\vec{\mathcal{U}}_r = \vec{0}$$

and

$$\vec{\mathcal{U}}_\ell^{\,T}\mathcal{A} = \vec{0} .$$

3.

4. In particular, one has

   1. the fact that

$$\mathcal{A}\begin{bmatrix} -\partial_t \\ \vec{\nabla} \end{bmatrix}\Lambda = \begin{bmatrix} -\nabla^2\partial_t - \partial_t\nabla^2 \\ 0 - \partial_t\vec{\nabla}\partial_t + \partial_t^2\vec{\nabla} \end{bmatrix}\Lambda = \begin{bmatrix} 0 \\ \vec{0} \end{bmatrix}$$

for all three-times differentiable scalar fields $\Lambda(t,\vec{x})$. Thus

$$\vec{\mathcal{U}}_r \equiv \begin{bmatrix} -\partial_t \\ \vec{\nabla} \end{bmatrix} \in \mathcal{N}(\mathcal{A}) .$$

The null space of $\mathcal{A}$ is therefore nontrivial and 1-dimensional at each $(t,\vec{x})$.

2. the fact that

$$\begin{bmatrix} \partial_t & \vec{\nabla}\cdot \end{bmatrix}\mathcal{A}\begin{bmatrix} \phi \\ \vec{A} \end{bmatrix} = -\partial_t\nabla^2\phi + \partial_t^2\vec{\nabla}\cdot\vec{A} + 0 + \partial_t\nabla^2\phi + \vec{\nabla}\cdot\partial_t^2\vec{A} = 0 ,$$

for all 4-vectors $\begin{bmatrix} \phi \\ \vec{A} \end{bmatrix}$. Thus

$$\vec{\mathcal{U}}_\ell^T \equiv \begin{bmatrix} \partial_t & \vec{\nabla}\cdot \end{bmatrix} \in \text{ left null space of } \mathcal{A} , \qquad (649)$$

or

$$\vec{\mathcal{U}}_\ell \in \mathcal{N}(\mathcal{A}^T) . \qquad (650)$$

The left null space of $\mathcal{A}$ is therefore also 1-dimensional at each $(t, \vec{x})$.

In light of the singular nature of $\mathcal{A}$, the four-component Maxwell waveequation

$$\mathcal{A}\begin{bmatrix} \phi \\ \vec{A} \end{bmatrix} = 4\pi \begin{bmatrix} \rho \\ \vec{J} \end{bmatrix} \tag{651}$$

has no solution unless the source $\begin{bmatrix} \rho \\ \vec{J} \end{bmatrix}$ also satisfies

$$\vec{\mathcal{U}}_\ell^T \begin{bmatrix} \rho \\ \vec{J} \end{bmatrix} = 0 \ .$$

This is the linear algebra way of expressing

$$\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0 \ , \tag{652}$$

the differential law of charge conservation. Thus Maxwell's equations apply if and only if the law of charge conservation holds. If charge conservation did not hold, then Maxwell's equations would be silent. They would not have a solution. Such silence is a mathematical way of expressing the fact that at its root theory is based on observation and established knowledge, and that arbitrary hypotheses must not contaminate the theoretical.

The Over determined System $A\vec{u} = \vec{b}$:

The linear algebra aspects of Maxwell's wave operator A are illustrated by the following problem from linear algebra:

Solve $A\vec{u} = \vec{b}$ for $\vec{u}$, under the stipulation that

$$
\begin{array}{ll}
A: & R^4 \longrightarrow R^4 \\
\vec{u}_r: & A\vec{u}_r = \vec{0} \quad \text{so that } \mathcal{N}(A) = span\{\vec{u}_r\} \\
\vec{u}_\ell^T: & \vec{u}_\ell^T A = \vec{0} \quad \text{so that } \mathcal{N}(A^T) = span\{\vec{u}_\ell\} \\
\vec{b}: & \vec{b} \in \mathcal{R}(A) \quad \text{so that } \vec{u}_\ell^T \vec{b} = 0
\end{array}
\tag{653}
$$

The fact that A is singular and $\vec{b}$ belongs to the range of A makes the system over-determined but consistent. This means that there are more equations than there are unknowns.
One solves the problem in two steps.

Step I:

Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be the set of eigenvectors having non-zero eigenvalues. Whatever A is, the task of finding three vectors that satisfy

$$
\left.
\begin{aligned}
A\vec{v}_1 c_1 &= \vec{v}_1 \lambda_1 c_1 \\
A\vec{v}_2 c_2 &= \vec{v}_2 \lambda_2 c_2 \\
A\vec{v}_3 c_3 &= \vec{v}_3 \lambda_3 c_3
\end{aligned}
\right\} \lambda_i \neq 0, \; c_i \text{ are scalars}
\tag{654}
$$

and

$$A\vec{u}_r c_4 = \vec{0} \ . \tag{655}$$

Being spanned by the three eigenvectors with non-zero eigenvalues, the range space of A,

$$\mathcal{R}(A) = span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \ ,$$

is well-determined. However, the scalars c_i are at this stage as yet undetermined.

Step II:

Continuing the development, recall that quite generally

[illegible]

$$\Leftrightarrow \vec{b} \in \mathcal{R}(A)$$

$$\Leftrightarrow \vec{b} = \vec{v}_1 b_1 + \vec{v}_2 b_2 + \vec{v}_3 b_3 \ , \tag{656}$$

and that if

$$\vec{u} \qquad = \vec{v}_1 c_1 + \vec{v}_2 c_2 + \vec{v}_3 c_3 + \vec{u}_4 c_4 \ ,$$

then

$$[illegible] \qquad = \ \vec{v}_1 \lambda_1 c_1 + \vec{v}_2 \lambda_2 c_2 + \vec{v}_3 \lambda_3 c_3 \ . \tag{657}$$

It is appropriate to alert the reader that in the ensuing section the vectors $\vec{v}_i$ and the eigenvalues λ_i become differential operators which act on scalar fields c_i and that the three subscript labels will refer to the TE, TM, and TEM eletromagnetic[66] vector potentials respectively.

Equating (6.56) and (6.57), one finds that the linear independence of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ implies the following equations for c_1, c_2, and c_3:

$$\lambda_1 c_1 \qquad = b_1 \longrightarrow c_1 = \frac{1}{\lambda_1}\, b_1 \tag{658}$$

$$\lambda_2 c_2 \qquad = b_2 \longrightarrow c_2 = \frac{1}{\lambda_2}\, b_2 \tag{659}$$

$$\lambda_3 c_3 \qquad = b_3 \longrightarrow c_3 = \frac{1}{\lambda_3}\, b_3 \tag{660}$$

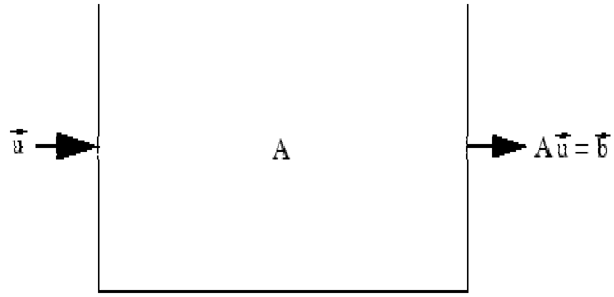
Consequently, the solution is

$$\vec{u} = \vec{v}_1 \frac{1}{\lambda_1}\, b_1 + \vec{v}_2 \frac{1}{\lambda_2}\, b_2 + \vec{v}_3 \frac{1}{\lambda_3}\, b_3 + \vec{u}_4 c_4$$

where $\vec{u}_4 c_4$ is an indeterminate multiple of the null space vector $\vec{u}_4$.

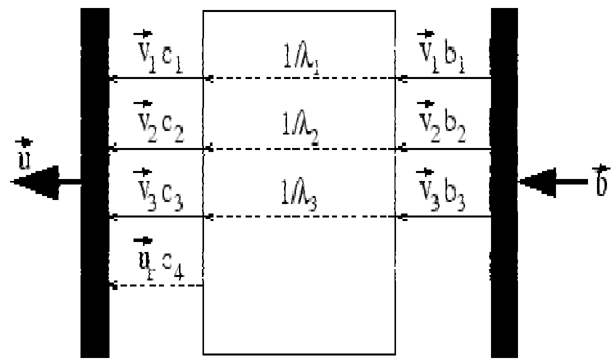
If one represents the stated problem $A\vec{u} = \vec{b}$ ($\vec{u}$ determines $\underline{\vec{b}}$) as an input-output process, as in Figure 6.3,

Figure 6.3: The matrix A defines an input-output process.



then its solution is represented by the inverse input-output process as in Figure 6.4.

Figure: The solution to $A\vec{u} = \vec{b}$ defines an inverse input-output process.



In general, the task of finding the eigenvectors of a 4 $\times$ 4 matrix can be a nontrivial task. However, given the fact that the solution to

$$\vec{u}_\ell^T \boldsymbol{A} = \vec{0}$$

is already known, one finds that the associated constraints on the eigenvectors,

$$\vec{u}_\ell^T \vec{v}_i = 0$$

make the task quite easy, if not trivial.

Maxwell Wave Equation (continued):

The above linear algebra two-step analysis of an over determined (but consistent) system $A\vec{u} = \vec{b}$ is an invaluable guide in solving Maxwell's wave equation

$$\mathcal{A}\begin{bmatrix}\phi\\ A_z\\ A_x\\ A_y\end{bmatrix} = 4\pi\begin{bmatrix}\rho\\ J_z\\ J_x\\ J_y\end{bmatrix}\begin{matrix}\left.\begin{matrix}\\ \\ \end{matrix}\right\}\text{longitudinal components}\\ \left.\begin{matrix}\\ \\ \end{matrix}\right\}\text{transverse components,}\end{matrix} \qquad \textbf{(661)}$$

Cylindrical Coordinates:

The benefits of the linear algebra viewpoint applied to Maxwell'sequations can be extended by inspection from rectilinear cartesian to cylindrical coordinates. This is because the four-dimensional coordinate system lends itself to being decomposed into two orthogonal sets of coordinate surfaces. For cylindricals these are spanned by

the *transverse* coordinates (r, θ) in the transverse plane, and

the *longitudinal* coordinates (z, t) in the longitudinal plane.

The transition from a rectilinear to a cylindrical coordinate frame is based on the replacement of the following symbols:

$$dx \longrightarrow dr \qquad ; \qquad dy \longrightarrow r\,d\theta \qquad (677)$$

$$\frac{\partial}{\partial x} \longrightarrow \frac{\partial}{\partial r} \qquad ; \qquad \frac{\partial}{\partial y} \longrightarrow \frac{1}{r}\frac{\partial}{\partial \theta} \qquad (678)$$

and

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \longrightarrow \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,. \qquad (679)$$

Such a replacement yields the vector field components relative to an orthonormal (o.n.) basis tangent to the coordinate lines. To emphasize this orthonormality, hats ($\hat{\_}$) are placed over the vector components.

This replacement is very powerful. It circumvents the necessity of having to repeat the previous calculations that went into exhibiting the individual components of Maxwell's TE , TM , and TEM systems of equations. We shall again take advantage of the power of this algorithm in the next section when we apply it to Maxwell's system relative to spherical coordinates.

Applying it within the context of cylindrical coordinates, one finds that the source and the vector potential four-vectors are as follows:

1. for a TE source

$$(\rho, \hat{J}_z, \hat{J}_r, \hat{J}_\theta) = \left(0, 0, \frac{1}{r}\frac{\partial S^{TE}}{\partial \theta}, -\frac{\partial S^{TE}}{\partial r}\right), \tag{680}$$

2. the solution to the Maxwell field equations has the form

$$(\phi, \hat{A}_z, \hat{A}_r, \hat{A}_\theta) = \left(0, 0, \frac{1}{r}\frac{\partial \Phi^{TE}}{\partial \theta}, -\frac{\partial \Phi^{TE}}{\partial r}\right); \tag{681}$$

for a TM source

$$(\rho, \hat{J}_z, \hat{J}_r, \hat{J}_\theta) = \left(-\frac{\partial S^{TM}}{\partial z}, \frac{\partial S^{TM}}{\partial t}, 0, 0\right), \tag{682}$$

the solution to the Maxwell field equations has the form

$$(\phi, \hat{A}_z, \hat{A}_r, \hat{A}_\theta) = \left(-\frac{\partial \Phi^{TM}}{\partial z}, \frac{\partial \Phi^{TM}}{\partial t}, 0, 0\right); \tag{683}$$

and

for a TEM source

$$(\rho, \hat{J}_z, \hat{J}_r, \hat{J}_\theta) = \left(-\frac{\partial J}{\partial t}, \frac{\partial J}{\partial z}, \frac{\partial I}{\partial r}, \frac{1}{r}\frac{\partial I}{\partial \theta}, \right), \tag{684}$$

3. the solution to the Maxwell field equations has the form

$$(\phi, \hat{A}_z, \hat{A}_r, \hat{A}_\theta) = \left(-\frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial z}, \frac{\partial \Psi}{\partial r}, \frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right). \tag{685}$$

Spherical Coordinates:

One of the chief virtues of the linear algebra viewpoint applied to Maxwell's equations is that it directs attention to the system's fundamental vector spaces and their properties. The easiest way to identify them in a computational way happens when the underlying coordinate system permits a 2+2 decomposition into what amounts to longitudinal and transverse surfaces. Spherical coordinates provide a nontrivial example of this. There a transverse surface is a sphere spanned by $\{\theta, \phi\}$, while the longitudinalcoordinates are (r, t).

The distinguishing feature of spherical coordinates, as compared to rectilinear or cylindrical coordinates, is that coordinate rectangles on successive transverse surfaces (nested spheres) are not congruent. Instead, they have areas that scale with the square of the radial distance from the origin. This scaling alters the representation of the divergence of a vector field and hence the Maxwell wave operator. Nevertheless, the eigenvalue method with its resulting TE-TM-TEM decomposition of the e.m. field readily accomodates these alterations.