Partial Differential Equations:

Linear algebra is the mathematical guide of choice for implementing the principle of unit-economy[61] applied to partial differential equations. The present chapter considers two kinds:

1. Single linear partial differential equationscorresponding to the linear system

$$A\vec{u} = \vec{0}\,.$$

However, instead of merely exhibiting generalsolutions to such a system, we shall take seriously the dictum which says that *a differential equation is never solved until ``boundary'' conditions have beenimposed on its solution.* As identified in the ensuing section, these conditions are not arbitrary. Instead, they fall into three archetypical classes determined bythe nature of the physical system which the partial differential equation conceptualizes.

1. Systems of pde's corresponding to an over-determined system

$$A\vec{u} = \vec{b}\,. \qquad (61)$$

The idea for solving it takes advantage of the*fundamental subspaces*[62]of A [#!StrangLinearAlgebra!#]. Let A be a 4×4 matrix having rank $\underline{3}$. Such a matrix, we recall, has a vector $\vec{u}_r$ which satisfies $A\vec{u}_r = \vec{0}$, or, to be more precise

$$A\vec{u}_r c_0 = \vec{0}\,, \qquad (62)$$

where c_0 is any non-zero scalar. Thus $\vec{u}_r$ spans A's one-dimensional nullspace

$$\mathcal{N}(A) = span\{\vec{u}_r\}\,.$$

This expresses the fact that the columnes of A are linearly dependent.

In addition we recall that the four rows of A are linearly dependent also, a fact which is expressed by the existence of a vector $\vec{u}_\ell$ which satisfies

$$\vec{u}_\ell^T A = \vec{0}\,, \tag{63}$$

and which therefore spans A's one-dimensional left nullspace

$$\mathcal{N}(A^T) = span\{\vec{u}_\ell^T\}\ .$$

In general there does not exist a solution to the over-determined system Eq.(6.1). However, a solution obviously does exist if and only if $\vec{b}$ satisfies

$$\vec{u}_\ell^T \vec{b} = 0\ .$$

Under such a circumstance there are infinitely many solutions, each one differing from any other merely by a multiple of the null vector $\vec{u}_r$. The most direct path towards these solutions is via eigenvectors.

One of them is, of course, the vector $\vec{u}_r$ in Eq.(6.2). The other three, which (for the A under consideration) are linearly independent, satisfy $A\vec{v}_i = \lambda_i \vec{v}_i$ with $\lambda_i \neq 0$, or, in the interest of greater precision (which is needed in Section 6.2.3),

$$A\vec{v}_1 c_1 = \lambda_1 \vec{v}_1 c_1 \tag{64}$$

$$A\vec{v}_2 c_2 = \lambda_2 \vec{v}_2 c_2 \tag{65}$$

$$A\vec{v}_3 c_3 = \lambda_3 \vec{v}_3 c_3 \tag{66}$$

where, like c_0, the c_i's are any non-zero scalars. Because of the simplicity of $\vec{u}_\ell^T$ for the A under consideration one can find the eigenvectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, and hence their eigenvalues, by a process of inspection. These vectors span the range of A,

$$\mathcal{R}(A) = span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}\ ,$$

and therefore determine those vectors $\vec{b}$ for which there exists a solution to Eq.(6.1). Such vectors belong to $\mathcal{R}$ and thus have the form

$$\vec{b} = \vec{v}_1 b_1 + \vec{v}_2 b_2 + \vec{v}_3 b_3 \ .$$

These eigenvectors also serve to represent the corresponding solution,

$$\vec{u} = \vec{u}_r c_0 + \vec{v}_1 c_1 + \vec{v}_2 c_2 + \vec{v}_3 c_3 \ .$$

This, the fact that $A\vec{u} = \vec{b}$, and the linear independence of the $\vec{v}_i$s imply that the scalars c_i satisfy the three equations

$$\lambda_1 c_1 = b_1 \tag{67}$$

$$\lambda_2 c_2 = b_2 \tag{68}$$

$$\lambda_3 c_3 = b_3 \ . \tag{69}$$

As expected, the contribution c_4 along the direction of the nullspace element is left indeterminate. These ideas fromlinear algebra and their application to solving a system, such as Eq.(6.1), can be extended to corresponding systems of partial differential equations. The Maxwell field equations, which we shall analalyze using linear algebra, is a premier example. In this extension the scalar entries of A and $\vec{u}_i$ get replaced by differential operators, the vectors $\vec{u}$ and $\vec{b}$ by vector fields, the scalars b_i and c_i by scalar fields, the eigenvalues λ_i by a second order wave operator, and the three Eqs.(6.7)- by three inhomogeneous scalar waveequations corresponding to what in physics and engineering are called

- *transverse electric* (TE),
- *transverse magnetic* (TM), and
- *transverse electric magnetic* (TEM),

modes respectively.

Single Partial Differential Equations: Their Origin

There are many phenomena in nature, which, even though occuring over finite regions of space and time, can be described in terms of properties that prevail at each point ofspace and time separately. This description originated with Newton, who with the aid of his differential calculus showed us how to grasp a global phenomenon, for example, the elliptic orbit of a planet, by means of a locally applied law, for example $F = ma$.

This manner of making nature comprehensible has been extended from the motion of single point particles to the behavior of other forms of matter and energy, be it in the form of gasses, fluids, light, heat, electricity, signals traveling along optical fibers and neurons, or even gravitation.

This extension consists of formulating or stating a partial differential equation governing the phenomenon, and then solving that differential equation for the purpose of predicting measurable properties of the phenomenon.

There exist many partial differential equations, but from the view point of mathematics, there are basically only *three types* of partial differential equations.

They are exemplified by

1. Laplaces equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0\,,$$

which governs electrostatic and magnetic fields as well as the velocity potential of an incompressible fluid, by

2. the wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

for electromagnetic or sound vibrations, and by

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

for the vibrations of a simple string, and by

3. the diffusion equation

$$\nabla^2\psi - \frac{1}{k}\,\frac{\partial\psi}{\partial t} = 0$$

for the temperature in three dimensional space and in time, or by

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{k}\,\frac{\partial\psi}{\partial t} = 0$$

for the temperature along a uniform rod.

Boundary Conditions of a Typical Partial Differential Equation in Two Dimensions:

For the purpose of simplicity, we shall start our considerationwith partial differential equations in only two variables and linear in the second derivatives. Such equations have the general form

$$A(x,y)\frac{\partial^2\psi}{\partial x^2} + 2B(x,y)\frac{\partial^2\psi}{\partial x\partial y} + C(x,y)\frac{\partial^2\psi}{\partial y^2} - \Phi\left(x,y,\psi,\frac{\partial\psi}{\partial x},\frac{\partial\psi}{\partial y}\right) .$$

Such an equation is called a quasilinear second order partial differential equation. If the expression Φ where linear in ψ, i.e., if

$$\Phi = D(x,y)\frac{\partial\psi}{\partial x} + E(x,y)\frac{\partial\psi}{\partial y} + F(x,y)\psi + G(x,y) ,$$

then the equation would be a linear p.d.e., but this need not be the case.

The equation has a nondenumerable infinity of solution. In order to single out a unique solution, the to-be-found function $\psi(x,y)$ must satisfy additional conditions. They are usually specified at the boundary of the domain of the p.d.e.

In three dimensional space, this boundary is a *surface*, but in our two dimensional case, we have a *boundary line* which can be specified by the parametrized curve

$$x = \xi(s)$$

$$y = \eta(s) \, ,$$

where s is the arclength parameter

$$s = \int ds = \int \sqrt{dx^2 + dy^2} \, .$$

The tangent to this curve has components

$$\left(\frac{d\xi}{ds} , \frac{d\eta}{ds} \right) \, .$$

They satisfy

$$\left(\frac{d\xi}{ds} \right)^2 + \left(\frac{d\eta}{ds} \right)^2 = 1 \, .$$

The normal to this boundary curve has components

$$\left(\frac{d\eta}{ds} , -\frac{d\xi}{ds} \right) = \vec{n} \, .$$

We assume that $\vec{n}$ points towards the *interior* of the domain where the solution is to be found. If this is not the case, we reverse the signs of the components of it.

The additional conditions which the to-be-found solution ψ is to satisfy are imposed at this boundary curve, and they are conditions on the partial derivatives and the value of the function ψ evaluated at the curve.

The boundary curve accomodates three important types ofboundary conditions.

1. *Dirichlet conditions:* $\psi(s)$ is specified at each point of the boundary.

2. *Neumann conditions:* $\frac{d\psi}{dn}(s) = \vec{n} \cdot \nabla\psi$, the normal componet of the graident of ψ is specified at each point of the boundary.

3. *Cauchy conditions:* $\psi(s)$ and $\frac{d\psi}{dn}(s)$ are specified at each point of the boundary. The parameter s is usually a time parameter. Consequently, Cauchy conditions are also called *intial value conditions* or*initial value data* or simply *Cauchy data*.

There exists also the *mixed Dirichlet-Neumann* conditions. They are intermediate between the Dirichlet and the Neumann boundary conditions, and they are given by

$$\alpha(s)\psi(s) + \beta(s)\frac{d\psi}{dn}(s) = f(s)\ .$$

Here $\alpha(s)$, $\beta(s)$, and $f(s)$ are understood to be given on the boundary.

We recall that in the theory of ordinary second order differential equations, a unique solution was obtained once the solution and its derivative were specified at a point. The generalization of this condition to partial differential equationsconsists of the Cauchy boundary conditions.

Consequently, we now inquire whether the solution of the partial differential equation is uniquely determined by specifying Cauchy boundary conditions on the boundary $(\xi(s), \eta(s))$ **.**

Cauchy Problem and Characteristics:

In order to compute the function $\psi(x,y)$ at points off theboundary curve, we resort to the Taylor series on twodimensions;

$$\psi(x,y) = \psi(\xi,\eta) + (x-\xi)\frac{\partial\psi}{\partial x} + (y-\eta)\frac{\partial\psi}{\partial y}$$

$$+\frac{1}{2!}\left[(x-\xi)^2\frac{\partial^2\psi}{\partial x^2} + 2(x-\xi)(y-\eta)\ \frac{\partial^2\psi}{\partial x\partial y} + (y-\xi)^2\frac{\partial^2\psi}{\partial y^2}\right] + \cdots .$$

Here the derivatives are to be evaluated on the boundary.

The problem we are confronted with is this:

Determine all partial derivatives, starting with the first partials on up from the given Cauchy boundary conditions, the givenboundary, and the given partial differential equation!

We shall do this first for the first derivatives.

From the Cauchy data we obtain two equations

$$\left.\begin{aligned}\frac{d\psi(s)}{dn} &= \frac{d\eta}{ds}\frac{\partial\psi}{\partial x} - \frac{d\xi}{ds}\frac{d\psi}{dy}\\ \frac{d\psi(s)}{ds} &= \frac{d\xi}{ds}\frac{\partial\psi}{\partial x} + \frac{d\eta}{ds}\frac{d\psi}{dy}\end{aligned}\right\}\quad \text{at}\quad (x,y) = (\xi(s),\eta(s)) \tag{610}$$

From these we obtain the first partial derivatives of ψ evaluates on the boundary

$$\begin{aligned}\left(\frac{\partial\psi}{\partial x}\right)_{(\xi,\eta)} &= \frac{d\psi(s)}{dn}\frac{d\eta}{ds} + \frac{d\psi}{ds}\frac{d\xi}{ds}\\ \left(\frac{\partial\psi}{\partial y}\right)_{(\xi,\eta)} &= -\frac{d\psi(s)}{dn}\frac{d\eta}{ds} + \frac{d\psi}{ds}\frac{d\xi}{ds}\end{aligned} \tag{611}$$

The procurement of the second derivatives is more interesting. We differentiate the (known) first derivatives along the boundary. Together with the given p.d.e. we have

$$\frac{d}{ds}\left(\frac{\partial\psi}{\partial x}\right) = \frac{d\xi}{ds}\frac{\partial^2\psi}{\partial x^2}+\frac{d\eta}{ds}\frac{\partial^2\psi}{\partial y\partial x}$$

$$\frac{d}{ds}\left(\frac{\partial\psi}{\partial y}\right) = \frac{d\xi}{ds}\frac{\partial^2\psi}{\partial x\partial y}+\frac{d\eta}{ds}\frac{\partial^2\psi}{\partial y^2}$$

$$\Phi = A\frac{\partial^2\psi}{\partial x^2} + 2B\frac{\partial^2\psi}{\partial x\partial y}+C\frac{\partial^2\psi}{\partial y^2}\,.$$

The left hand sides of these three equations are known along the whole boundary. So are the coefficients of the three unknown partial derivatives on the right hand side. One can solve for these partial derivatives unless

$$\begin{vmatrix} \frac{d\xi}{ds} & \frac{d\eta}{ds} & 0 \\ 0 & \frac{d\xi}{ds} & \frac{d\eta}{ds} \\ A & 2B & C \end{vmatrix} = 0$$

or

$$A\left(\frac{d\eta}{ds}\right)^2 - 2B\frac{d\eta}{ds}\,\frac{d\xi}{ds} + C\left(\frac{d\xi}{ds}\right)^2 = 0\,.$$

If this determinant does not vanish, one can solve for the secondderivatives evaluated on the boundary. Differentiating along theboundary yields

$$\frac{d}{ds}\psi_{xx} = \frac{d\xi}{ds}\psi_{xxx}+\frac{d\eta}{ds}\psi_{yxx}$$

$$\frac{d}{ds}\psi_{xy} = \frac{d\xi}{ds}\psi_{xxy}+\frac{d\eta}{ds}\psi_{yxy}$$

$$\Phi_x + \cdots = A\psi_{xxx} + 2B\psi_{xyx}+C\psi_{xyy}\,.$$

Subscripts refer to partial derivatives. The last equation was obtained differentiating the given p.d.e. with respect to x . The left hand side contains only lower order derivatives, which are known on the boundary.

We see that one can solve for

$$\psi_{xxx}\,,\ \psi_{yxx}\,,\ \psi_{xyy}$$

on the boundary unless the determinant, the same one as before, vanishes. It is evident that one can continue the process of solving for the other higher order derivatives, provided the determinant of the system does not vanish. We are led to the conclusion that one can expand $\psi(x,y)$ in a Taylor series at every point of the boundary and that the coefficients of the series are uniquely determined by the Cauchy boundary conditions on the given boundary.

We must now examine the vanishing of the system determinant

$$A(x,y)\left(\frac{dy}{ds}\right)^2 - 2B(x,y)\frac{dy}{ds}\ \frac{dx}{ds} + C(x,y)\left(\frac{dx}{ds}\right)^2 = 0 \qquad (612)$$

at every point of the domain of the partial differential equation.

Depending on the coefficients A , B , and C , this quadratic form determines two characteristic curves, $\lambda(x,y)=const.$ and $\mu(x,y)=const.$, through each point (x,y) . We distinguish between three cases:

1. $AC-B^2>0$: *elliptic type* in which the two characteristics λ and μ are complex conjugates of each other.
2. $AC-B^2<0$: *hyperbolic type* in which case for each (x,y) the characteristics λ and μ are real. They sre two curves intersecting at (x,y) . As one varies (x,y) one obtains two distinct families.

3. $AC - B^2 = 0$: *parabolic type* in which there is only one family of characteristics.

These three cases imply three different types of differentialequations. By utilizing the characteristic, one can introduce newcoordinates relative to which a differential equation of each type assumes a standard normal form. Let the new coordinate surfaces be

$$\lambda(x,y) = \text{const} \qquad \mu(x,y) = \text{const}\,.$$

Then the coordinate transformation

$$u + iv = \lambda \quad \text{and} \quad u - iv = \mu$$

yields a normal form of the *elliptic type*,

$$\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2} = \Phi\left(u, v, \psi, \frac{\partial\psi}{\partial u},\ \frac{\partial\psi}{\partial v}\right)\,.$$

By contrast the coordinate transformation

$$\lambda = \lambda(x,y) \quad \text{and} \quad \mu = \mu(x,y)$$

yields a normal form of the *hyperbolic type*,

$$\frac{\partial^2\psi}{\partial\lambda\partial\mu} = \Phi\left(\lambda, \mu, \psi, \frac{\partial\psi}{\partial\lambda},\ \frac{\partial\psi}{\partial\mu}\right)\,. \qquad (613)$$

Finally, the coordinate transformation

$$\lambda = \lambda(x,y) = \mu(x,y), \quad x = x$$

yields a normal form of the *parabolic type*,

$$\frac{\partial^2\psi}{\partial\lambda^2} = \Phi\left(x, \lambda, \psi, \frac{\partial\psi}{\partial x},\ \frac{\partial\psi}{\partial\lambda}\right)\,.$$

We recognize that *elliptic* partial differential equations express an*equilibrium* or a static potential phenomenon.

By introducing the standard coordinates

$$t = \lambda + \mu \qquad \text{and} \qquad z = \lambda - \mu$$

in terms of which

$$\lambda = \frac{1}{2}(t + z) \qquad \text{and} \qquad \mu = \frac{1}{2}(t - z)\ ,$$

one finds that

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial z^2} = \Phi' \left(t, z, \psi, \frac{\partial \psi}{\partial t}\ , \ \frac{\partial \psi}{\partial z} \right)\ ,$$

the wave equation of a general vibrating string. We, therefore, recognize that a *hyperbolic* p.d. equation expresses the phenomenon of a *propagating wave* or *disturbance*.

Finally, a parabolic p.d. equation expresses a diffusion process. In fact, the two dimensional *Laplace equation*, the *equation for a vibrating sting*, and the *heat conduction equation* are the simplest possible examples of *elliptic*, *hyperoblic*, and *parabolic equations*.

Hyperbolic Equations:

The quadratic form, Eq.(6.12), determined by the coefficients A , B , and C of the given p.d.e. can be factored into two ordinarydifferential equation

$$A\,dy = \left(B + \sqrt{B^2 - AC}\right) dx \qquad \text{and} \qquad A\,dy = \left(B - \sqrt{B^2 - AC}\right) dx\,.$$

These are the equations for the *two families of characteristic curves*of the given p.d.e.

Their significance, we recall, is this: if the boundary line coincides with one of them, then specifying Cauchy data on it *will not* yield a unique solution. If, however, the boundary line intersects each family only once, then the Cauchy data *will* yields a unique solution.

This point becomes particularly transparent if one introduces the curvilinear coordinates λ and μ relative to which the given p.d.e. assumes its standard form, Eq.(>6.13). We shall consider the hyperbolic case by assuming that

$$B^2(x,y) - A(x,y)C(x,y) > 0$$

throughout the (x,y) domain.

We shall demand the new coordinates λ and μ - the*characteristic coordinates* - have the property that their isograms (``loci of points of constant values") contain the characteristic lines $\{x(s), y(s)\}$, i.e.,

$$\lambda(x(s), y(s)) = \text{const} \quad \text{and} \quad \mu(x(s), y(s)) = \text{const}$$

for all s. This implies that

$$\lambda_x \frac{dx}{ds} + \lambda_y \frac{dy}{ds} = 0 \quad \text{and} \quad \mu_x \frac{dx}{ds} + \mu_y \frac{dy}{ds} = 0$$

where, as usual

$$\lambda_x = \frac{\partial \lambda}{\partial x}, \text{etc.}$$

Substituting these equations into Eq.(6.12), the equation for the characteristic directions, one obtains

$$A\left(\frac{\partial \lambda}{\partial x}\right)^2 + 2B\frac{\partial \lambda}{\partial x}\frac{\delta \lambda}{\delta y} + C\left(\frac{\delta \lambda}{\delta y}\right)^2 = 0\,. \qquad (614)$$

An equation with the same coefficients is obtained for the otherfunction $\mu(x,y)$. The two solutions $\lambda(x,y)$ and $\mu(x,y)$ are real valued functions. Their isograms, the *characteristics* of the hyperbolic equation, give us the new curvilinear coordinate system

$$\lambda = \lambda(x,y) \qquad \mu = \mu(x,y)\ .$$

The partial derivatives of the given differential equation are now as follows

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial\lambda^2}(\lambda_x)^2 + 2\frac{\partial^2\psi}{\partial\lambda\partial\mu}\lambda_x\mu_x + \frac{\partial^2\psi}{\partial\mu^2}(\mu_x)^2 + \cdots \quad \frac{\partial^2\psi}{\partial x\partial y} =$$

$$\frac{\partial^2\psi}{\partial\lambda^2}\lambda_x\lambda_y + \frac{\partial^2\psi}{\partial\lambda\partial\mu}(\lambda_x\mu_y + \mu_x\lambda_y) + \frac{\partial^2\psi}{\partial\mu^2}\mu_x\mu_y + \cdots \quad \frac{\partial^2\psi}{\partial y^2} =$$

$$\frac{\partial^2\psi}{\partial\lambda^2}\lambda_y^2 + 2\frac{\partial^2\psi}{\partial\lambda\partial\mu}\lambda_y\mu_y + \frac{\partial^2\psi}{\partial\mu^2}\mu_y^2 + \cdots\ .$$

Here $+\cdots$ refers to additional terms involving only the first partial derivatives of ψ. Inserting these expressions into the given p.d.equation, one obtains

$$+\ [A\mu_x^2 + 2B\mu_x\mu_y + C\mu_y^2]\frac{\partial^2\psi}{\partial\mu^2}\ [A\lambda_x^2 - 2B\lambda_x\lambda_y + C\lambda_y^2]\frac{\partial^2\psi}{\partial\lambda^2} +$$

$$[2A\lambda_x\mu_x + B(\lambda_x\mu_y + \mu_x\lambda_y) + 2C\lambda_y\mu_y]\frac{\partial^2\psi}{\partial\lambda\partial\mu} = \Phi'\left(\lambda,\mu,\psi,\frac{\partial\psi}{\partial\lambda},\ \frac{\partial\psi}{\partial\mu}\right)\ . \qquad (615)$$

It follows from Equation 6.14 that the coefficients of $\psi_{\lambda\lambda}$ and $\psi_{\mu\mu}$ vanish. Solving for $\frac{\partial^2\psi}{\partial\lambda\partial\mu}$ yields Equation > 6.13, the hyperbolicequation in normal form.

The coordinates λ and μ, whose surfaces contain the characteristic lines, are called the *characteristic coordinates* or *null coordinates* of the hyperbolic equation.

These coordinates are important for at least two reasons. First of all, they are boundaries across which a solution can be nonanalytic. If $\lambda(x,y)=\lambda_0$ is one of the isograms (``locus of points where λ has constant value'') of the solution to Eq.(6.14), then the first term of the p.d. Eq.(6.15)

$$[A\lambda_x^2+2B\lambda_x\lambda_y+C\lambda_y^2]\frac{\partial^2\psi}{\partial\lambda^2}=finite$$

even if $\frac{\partial^2\psi}{\partial\lambda^2}\to\infty$ as $\lambda\to\lambda_0$. In other words, *there aresolutions to Eq.(6.15) for which the first derivative* $\frac{\partial\psi}{\partial\lambda}$ *has a discontinuity across the characteristic* $\lambda(x,y)=\lambda_0$. Similarly, there exist solutions to Eq.(6.15) whose first derivative $\frac{\partial\psi}{\partial\mu}$ has a discontinuity across $\mu(x,y)=\mu_0$ whenever $\mu(x,y)$ satisfies Eq.(6.14) with λ replaced by μ.

Secondly, these coordinates depict the history of a moving disturbance. The simple string illustrates the issue involved.

Example: The Simple string The governing equation is

$$\frac{\partial^2\psi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=0$$

Its characteristic coordinates are the ``retarded'' and the ``advanced'' times

$$\lambda=ct-z \quad \text{and} \quad \mu=z+ct$$

and its normal form is

$$\frac{\partial^2\psi}{\partial\lambda\partial\mu}=0\,.$$

The solution is

$\psi = f(\lambda) + g(\mu)$

where f and g are *any* functions of λ and μ .

Next consider the initial value data at $t = 0$:

$\psi_0(z) \equiv \psi(t=0, z) \;=\; f(-z) - g(z)$ "initial amplituce" $V_0(z) \equiv \left.\dfrac{\partial\psi(t,z)}{\partial t}\right|_{t=0} =$

$\left.\dfrac{\partial\lambda}{\partial t}\dfrac{\partial\psi}{\partial\lambda}\right|_{\lambda=-z} + \left.\dfrac{\partial\mu}{\partial t}\dfrac{\partial\psi}{\partial\mu}\right|_{\mu=z}$ "initial velocity" $= cf'(-z) + cg'(z)$.

These equations imply

$$f(\lambda) \quad = \frac{1}{2}\psi_0(-\lambda) + \frac{1}{2c}\int_0^{-\lambda} V_0(z')dz'$$

$$g(\mu) \quad = \frac{1}{2}\psi_0(\mu) + \frac{1}{2c}\int_0^{\mu} V_0(z')dz' .$$

Consider the intersection of the two families of characteristics with the boundary line $t = 0$ as in the figure below.