**GATE Material for Instrumentation Engineering**

**Engineering Mathematics**

* Linear Algebra
* Calculus
* Complex Analysis
* Vector Calculus
* Differential Equations
* Probability - I
* Probability (continued)
* Numerical Methods  
  (above topics are successfully completed in mathematics part)

**TRANSFORM THEORY**

* [Fourier Transform](http://onestopgate.com/gate-study-material/instrumentation/fourier-transform.asp)
* [Laplace Transform](http://onestopgate.com/gate-study-material/instrumentation/laplace-transform.asp)
* [Z-Transform](http://onestopgate.com/gate-study-material/instrumentation/z-transform.asp)

[**Fourier Transform**](http://onestopgate.com/gate-study-material/instrumentation/fourier-transform.asp)**:**

The Fourier transform is a generalization of the complex Fourier series in the limitas L->infty. Replace the discrete A_n with the continuous F(k)dk while letting n/L->k. Then change the sum to an integral, and the equations become

f(x) = int_(-infty)^inftyF(k)e^(2piikx)dk

(1)

F(k) = int_(-infty)^inftyf(x)e^(-2piikx)dx.

(2)

Here,

F(k) = ![F_x[f(x)](k)](data:image/gif;base64,)

(3)

http://onestopgate.com/images/resistors/fourier-transform/Inline14.gif = int_(-infty)^inftyf(x)e^(-2piikx)dx

(4)

is called the *forward* (-i) Fourier transform, and

f(x) = ![F_k^(-1)[F(k)](x)](data:image/gif;base64,)

(5)

http://onestopgate.com/images/resistors/fourier-transform/Inline21.gif = int_(-infty)^inftyF(k)e^(2piikx)dk

(6)

is called the *inverse* (+i) Fourier transform. The notation ![F_x[f(x)](k)](data:image/gif;base64,) is introduced in Trott (2004, p. xxxiv), and f^^(k) and f^_(x) are sometimes also used to denote the Fourier transform and inverse Fourier transform, respectively (Krantz 1999, p. 202).

Note that some authors (especially physicists) prefer to write the transform in terms of angular frequency omega=2pinu instead of the oscillation frequency nu. However, this destroys the symmetry, resulting in the transform pair

H(omega) = F[h(t)]

(7)

http://onestopgate.com/images/resistors/fourier-transform/Inline33.gif = int_(-infty)^inftyh(t)e^(-iomegat)dt

(8)

h(t) = F^(-1)[H(omega)]

(9)

http://onestopgate.com/images/resistors/fourier-transform/Inline39.gif = 1/(2pi)int_(-infty)^inftyH(omega)e^(iomegat)domega.

(10)

To restore the symmetry of the transforms, the convention

g(y) = F[f(t)]

(11)

http://onestopgate.com/images/resistors/fourier-transform/Inline45.gif = 1/(sqrt(2pi))int_(-infty)^inftyf(t)e^(-iyt)dt

(12)

f(t) = F^(-1)[g(y)]

(13)

http://onestopgate.com/images/resistors/fourier-transform/Inline51.gif = 1/(sqrt(2pi))int_(-infty)^inftyg(y)e^(iyt)dy

(14)

is sometimes used (Mathews and Walker 1970, p. 102).

In general, the Fourier transform pair may be defined using two arbitrary constantsa and b as

F(omega) = sqrt((|b|)/((2pi)^(1-a)))int_(-infty)^inftyf(t)e^(ibomegat)dt

(15)

f(t) = sqrt((|b|)/((2pi)^(1+a)))int_(-infty)^inftyF(omega)e^(-ibomegat)domega.

(16)

The Fourier transform F(k) of a function f(x) is implemented asFourierTransform[*f*, *x*, *k*], and different choices of a and b can be used by passing the optional FourierParameters-> {*a*, *b*} option. By default, *Mathematica*takes FourierParameters as (0,1). Unfortunately, a number of other conventions are in widespread use. For example, (0,1) is used in modern physics, (1,-1) is used in pure mathematics and systems engineering, (1,1) is used in probability theory for the computation of the characteristic function, (-1,1) is used in classical physics, and (0,-2pi) is used in signal processing. In this work, following Bracewell (1999, pp. 6-7), *it is always assumed that*a=0*and*b=-2pi*unless otherwise stated.* This choice often results in greatly simplified transforms of common functions such as 1, cos(2pik_0x), etc.

Since any function can be split up into even and odd portions E(x) and O(x),

f(x) = 1/2[f(x)+f(-x)]+1/2[f(x)-f(-x)]

(17)

http://onestopgate.com/images/resistors/fourier-transform/Inline82.gif = E(x)+O(x),

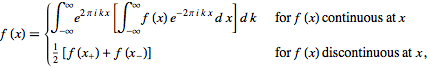
(18)

a Fourier transform can always be expressed in terms of the Fourier cosinetransform and Fourier sine transform as

![ F_x[f(x)](k)=int_(-infty)^inftyE(x)cos(2pikx)dx-iint_(-infty)^inftyO(x)sin(2pikx)dx. ](data:image/gif;base64,)

(19)

A function f(x) has a forward and inverse Fourier transform such that



(20)

provided that

1. int_(-infty)^infty|f(x)|dx exists.

2. There are a finite number of discontinuities.

3. The function has bounded variation. A sufficient weaker condition is fulfillment of the Lipschitz condition

(Ramirez 1985, p. 29). The smoother a function (i.e., the larger the number ofcontinuous derivatives), the more compact its Fourier transform.

The Fourier transform is linear, since if f(x) and g(x) have Fourier transforms F(k) and G(k), then

|  |  |  |  |
| --- | --- | --- | --- |
| int[af(x)+bg(x)]e^(-2piikx)dx | = | aint_(-infty)^inftyf(x)e^(-2piikx)dx+bint_(-infty)^inftyg(x)e^(-2piikx)dx | (21) |
| http://onestopgate.com/images/resistors/fourier-transform/Inline94.gif | = | aF(k)+bG(k). | (22) |

Therefore,

|  |  |  |  |
| --- | --- | --- | --- |
| F[af(x)+bg(x)] | = | aF[f(x)]+bF[g(x)] | (23) |
| http://onestopgate.com/images/resistors/fourier-transform/Inline100.gif | = | aF(k)+bG(k). | (24) |

The Fourier transform is also symmetric since ![F(k)=F_x[f(x)](k)](data:image/gif;base64,) implies![F(-k)=F_x[f(-x)](k)](data:image/gif;base64,).

Let f*g denote the convolution, then the transforms of convolutions of functions have particularly nice transforms,

|  |  |  |  |
| --- | --- | --- | --- |
| F[f*g] | = | F[f]F[g] | (25) |
| F[fg] | = | F[f]*F[g] | (26) |
| F^(-1)[F(f)F(g)] | = | f*g | (27) |
| F^(-1)[F(f)*F(g)] | = | fg. | (28) |

The first of these is derived as follows:

|  |  |  |  |
| --- | --- | --- | --- |
| F[f*g] | = | int_(-infty)^inftyint_(-infty)^inftye^(-2piikx)f(x^')g(x-x^')dx^'dx | (29) |
| http://onestopgate.com/images/resistors/fourier-transform/Inline121.gif | = | int_(-infty)^inftyint_(-infty)^infty[e^(-2piikx^')f(x^')dx^'][e^(-2piik(x-x^'))g(x-x^')dx] | (30) |
| http://onestopgate.com/images/resistors/fourier-transform/Inline124.gif | = | [int_(-infty)^inftye^(-2piikx^')f(x^')dx^'][int_(-infty)^inftye^(-2piikx^(''))g(x^(''))dx^('')] | (31) |
| http://onestopgate.com/images/resistors/fourier-transform/Inline127.gif | = | F[f]F[g], | (32) |

where x^('')=x-x^'.

There is also a somewhat surprising and extremely important relationship between the autocorrelation and the Fourier transform known as the Wiener-Khinchin theorem. Let ![F_x[f(x)](k)=F(k)](data:image/gif;base64,), and f^_ denote the complex conjugate of f, then the Fourier transform of the absolute square of F(k) is given by

|  |  |
| --- | --- |
| F_k[|F(k)|^2](x)=int_(-infty)^inftyf^_(tau)f(tau+x)dtau. | (33) |

The Fourier transform of a derivative f^'(x) of a function f(x) is simply related to the transform of the function f(x) itself. Consider

|  |  |
| --- | --- |
| F_x[f^'(x)](k)=int_(-infty)^inftyf^'(x)e^(-2piikx)dx. | (34) |

Now use integration by parts

|  |  |
| --- | --- |
| intvdu=[uv]-intudv | (35) |

with

|  |  |  |  |
| --- | --- | --- | --- |
| du | = | f^'(x)dx | (36) |
| v | = | e^(-2piikx) | (37) |

and

|  |  |  |  |
| --- | --- | --- | --- |
| u | = | f(x) | (38) |
| dv | = | -2piike^(-2piikx)dx, | (39) |

then

|  |  |
| --- | --- |
| F_x[f^'(x)](k)=[f(x)e^(-2piikx)]_(-infty)^infty-int_(-infty)^inftyf(x)(-2piike^(-2piikx)dx). | (40) |

The first term consists of an oscillating function times f(x). But if the function is bounded so that

|  |  |
| --- | --- |
| lim_(x->+/-infty)f(x)=0 | (41) |

(as any physically significant signal must be), then the term vanishes, leaving

|  |  |  |  |
| --- | --- | --- | --- |
| F_x[f^'(x)](k) | = | 2piikint_(-infty)^inftyf(x)e^(-2piikx)dx | (42) |
| http://onestopgate.com/images/resistors/fourier-transform/Inline154.gif | = | 2piikF_x[f(x)](k). | (43) |

This process can be iterated for the nth derivative to yield

|  |  |
| --- | --- |
| F_x[f^((n))(x)](k)=(2piik)^nF_x[f(x)](k). | (44) |

The important modulation theorem of Fourier transforms allows ![F_x[cos(2pik_0x)f(x)](k)](data:image/gif;base64,) to be expressed in terms of ![F_x[f(x)](k)=F(k)](data:image/gif;base64,) as follows,

|  |  |  |  |
| --- | --- | --- | --- |
| F_x[cos(2pik_0x)f(x)](k) | = | int_(-infty)^inftyf(x)cos(2pik_0x)e^(-2piikx)dx | (45) |
| http://onestopgate.com/images/resistors/fourier-transform/Inline163.gif | = | 1/2int_(-infty)^inftyf(x)e^(2piik_0x)e^(-2piikx)dx+1/2int_(-infty)^inftyf(x)e^(-2piik_0x)e^(-2piikx)dx | (46) |
| http://onestopgate.com/images/resistors/fourier-transform/Inline166.gif | = | 1/2int_(-infty)^inftyf(x)e^(-2pii(k-k_0)x)dx+1/2int_(-infty)^inftyf(x)e^(-2pii(k+k_0)x)dx | (47) |
| http://onestopgate.com/images/resistors/fourier-transform/Inline169.gif | = | 1/2[F(k-k_0)+F(k+k_0)]. | (48) |

Since the derivative of the Fourier transform is given by

|  |  |
| --- | --- |
| F^'(k)=d/(dk)F_x[f(x)](k)=int_(-infty)^infty(-2piix)f(x)e^(-2piikx)dx, | (49) |

it follows that

|  |  |
| --- | --- |
| F^'(0)=-2piiint_(-infty)^inftyxf(x)dx. | (50) |

Iterating gives the general formula

|  |  |  |  |
| --- | --- | --- | --- |
| mu_n | = | int_(-infty)^inftyx^nf(x)dx | (51) |
| http://onestopgate.com/images/resistors/fourier-transform/Inline175.gif | = | (F^((n))(0))/((-2pii)^n). | (52) |