DIFFERENTIAL EQUATIONS

EXERCISE – 11(D)

I. Solve the following differential equations.

1. $\frac{dy}{dx} = -\frac{12x + 5y - 9}{5x + 2y - 4}$

Sol. From given differential equation

$b = -5, a = 5 \Rightarrow b = -a$

$(5x + 2y - 4)dy = -(12x + 5y - 9)dx$

$(5x + 2y - 4)dy + (12x + 5y - 9)dx = 0$

$5(x\,dy + y\,dx) + 2y\,dy - 4\,dy + 12x\,dx - 9\,dx = 0$

Integrating $5xy + y^2 - 4y + 6x^2 - 9x = c$.

2. $\frac{dy}{dx} = \frac{-3x - 2y + 5}{2x + 3y + 5}$

Sol. From given differential equation

$b = -2, a = 2 \Rightarrow b = -a$

$(2x + 3y + 5)dy = (-3x - 2y + 5)dx$

$2x\,dy + 3y\,dy + 5dy = -3x\,dx - 2y\,dx + 5\,dx$

$2x\,dy + 3y\,dy + 5dy + 3x\,dx - 2y\,dx + 5\,dx = 0$

Integrating

$$2xy + \frac{3}{2}y^2 + \frac{3}{2}x^2 + 5y - 5x = c$$

$$4xy + 3y^2 + 3x^2 - 10x + 10y = 2c = c'$$

Solution is

$4xy + 3(x^2 + y^2) - 10(x - y) = c$.

3. $\frac{dy}{dx} = \frac{-3x - 2y + 5}{2x + 3y - 5}$

Ans: $3x^2 + 4xy + 3y^2 - 10x - 10y = c$

which is the required solution.

4. $2(x - 3y + 1)\frac{dy}{dx} = 4x - 2y + 1$

Ans: $2xy - 3y^2 - 2x^2 + 2y - x = c$.

5. $\frac{dy}{dx}=\frac{x-y+2}{x+y-1}$

Sol. $2xy+y^2-x^2-2y-4x=2c=c'$

6. $\frac{dy}{dx}=\frac{2x-y+1}{x+2y-3}$

Ans: xy + y^2 – x^2 – 3x – x = c

II. Solve the following differential equations.

1. (2x + 2y + 3) $\frac{dy}{dx}$ = x + y + 1

Sol. $\frac{dy}{dx}=\frac{x+y+1}{2x+2y+3}=\frac{(x+y)+1}{2(x+y)+3}$

Let v = x + y so that $\frac{dv}{dx}=1+\frac{dy}{dx}$

$\frac{dv}{dx}=1+\frac{v+1}{2v+3}=\frac{2v+3+v+1}{2v+3}=\frac{3v+4}{2v+3}$

$\frac{2v+3}{3v+4}dv=dx$

$\frac{2}{3}\int dv+\frac{1}{9}\int\frac{3\cdot dv}{3v+4}=\int dx$

$\frac{2}{3}v+\frac{1}{9}\log(3v+4)=x+c$

$\Rightarrow$ 6v + log(3v + 4) = 9x + 9c

$\Rightarrow$ 6(x + y) + log[3(x + y) + 4] = 9x + c

Hence solution is log(3x + 3y + 4) = 3x – 6y + c

2. $\frac{dy}{dx}=\frac{4x+6y+5}{2x+3y+4}$

Sol. $\frac{dy}{dx}=\frac{4x+6y+5}{2x+3y+4}=\frac{2(2x+3y)+5}{2x+3y+4}$

Let v = 2x + 3y

$\frac{dv}{dx}=2+3\frac{dy}{dx}\Rightarrow\frac{dv}{dx}=2+\frac{3(2v+5)}{v+4}$

$=\frac{2v+8+6v+15}{v+4}=\frac{8v+23}{v+4}$

$\frac{v+4}{8v+23}dv=dx$

$$\frac{1}{8}\int dv + \frac{9}{8}\int \frac{dv}{8v+23} = \int dx$$

$$\frac{1}{8}v + \frac{9}{64}\log(8v+23) = x + c$$

$\Rightarrow 8v + 9\log(8v + 23) = 64x + 64c$

$8(2x + 3y) - 64x + 9\log(16x + 24y + 23) = c'$

$$\Rightarrow 2x + 3y - 8x + \frac{9}{8}\log(16x + 24y + 23) = c''$$

$$3x - 6x + \frac{9}{8}\log(16x + 24y + 23) = c''$$

Hence solution is:

$$y - 2x + \frac{3}{8}\log(16x + 24y + 23) = k$$

3. (2x + y + 1)dx + (4x + 2y – 1)dy = 0

Sol. $\frac{dy}{dx} = -\frac{2x + y + 1}{4x + 2y - 1}$

$\Rightarrow a_1 = 2, b_1 = 1, a_2 = 4, b_2 = 2$

$$\frac{a_1}{a_2} = \frac{2}{4} = \frac{1}{2} = \frac{b_1}{b_2}$$

Let $2x + y = v$ so that $\frac{dv}{dx} = 2 + \frac{dy}{dx}$

$$\frac{dv}{dx} = 2 - \frac{v+1}{2v-1} = \frac{4v - 2 - v - 1}{2v-1} = \frac{3(v-1)}{2v-1}$$

$$\frac{2v-1}{3(v-1)}dv = dx \Rightarrow \frac{2v-1}{v-1}dv = 3dx$$

$$\int\left(2 + \frac{1}{v-1}\right)dv = 3\int dx$$

$2v + \log(v - 1) = 3x + c$

$2v - 3x + \log(v - 1) = c$

$2(2x + y) - 3x + \log(2x + y - 1) = c$

$4x + 2y - 3x + \log(2x + y - 1) = c$

Solution is $x + 2y + \log(2x + y - 1) = c$

4. $\frac{dy}{dx}=\frac{2y+x+1}{2x+4y+3}$

Sol. $\frac{dy}{dx}=\frac{2y+x+1}{2x+4y+3}$

Let v = x + 2y so that $\frac{dv}{dx}=1+\frac{2dy}{dx}$

$\frac{dv}{dx}=1+\frac{2(v+1)}{2v+3}=\frac{2v+3+2v+2}{2v+3}=\frac{4v+5}{2v+3}$

$\frac{2v+3}{4v+5}dv=dx \Rightarrow \int\left(\frac{1}{2}+\frac{1}{2(4v+5)}\right)dv=\int dx$

$\frac{1}{2}v+\frac{1}{2}\cdot\frac{1}{4}\log(4v+5)=x+c$

$\Rightarrow 4v+\log(4v+5)=8x+8c$

$\Rightarrow 4(x+2y)-8x+\log[4(x+2y)+5]=c'$

4x + 8y – 8x + log(4x + 8y + 5) = c′

$\Rightarrow$ 8y – 4x + log(4x + 8y + 5) = c′ which is solution of the given differential equation.

5. (x + y – 1)dy = (x + y + 1)dx

Ans:(x – y) + log (x + y) = c

III. Solve the following differential equations.

1. $\frac{dy}{dx}=\frac{3y-7x+7}{3x-7y-3}$

Sol.

$\Rightarrow a_1=-7, b_1=3, a_2=3, b_2=-7$

$\frac{a_1}{a_2}=\frac{-7}{3}, \frac{b_1}{b_2}=\frac{3}{-7} \quad \therefore \frac{a_1}{a_2}\neq\frac{b_1}{b_2}$

Let x = x + h, y = y + k

Where $3k-7h+7=0$ and$3h-7k-3=0$ and $\frac{dy}{dx}=\frac{dY}{dX}$

Solving these equations,

h = 0 and k = 1

$\frac{dy}{dx}=\frac{3y-7x}{3x-7y}$ which is a homogeneous differential equation.

Put y = vx $\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x(3v-7)}{x(3-7v)}$$

$$x\frac{dv}{dx} = \frac{3v-7}{3-7v} - v = \frac{3v-7-3v+7v^2}{3-7v}$$

$$= \frac{7v^2-7}{3-7v} = \frac{7v^2-7}{3-7v}$$

$$\frac{3-7v}{7v^2-7}dv = \frac{dx}{x}$$

$$\int\frac{3}{7v^2-7}dv - \int\frac{7vdv}{7v^2-7} = \int\frac{dx}{x}$$

$$\ln\ x = \frac{3}{14}\ln\left|\frac{v-1}{v+1}\right| - \frac{1}{2}\ln|v^2-1|14\ \log\ x\text{–}\log\ c$$

$$x = 3\log\left|\frac{v-1}{v+1}\right| - 7\log|v^2-1| \Rightarrow 14\ln x - \ln c$$

$$= 3\ln(v-1) - 3\ln(v+1) - 7\ln(v+1) - 7\ln(v-1)$$

$$14\ln x - \ln c = -10\ln(v+1) - 4\ln(v-1)$$

$$\ln(v+1)^5 + \ln(v-1)^2 + \ln x^7 = \ln c$$

$$(v+1)^5 \cdot (v-1)^2 \cdot x^7 = c$$

$$\left(\frac{y}{x}+1\right)^5\left(\frac{y}{x}-1\right)^2 x^7 = c$$

$$(y-x)^2(y+x)^5 = c$$

$$[y-(x-1)]^2(y+x-1)^5 = c$$

Solution is $[y-x+1)]^2(y+x-1)^5 = c$

2. $\frac{dy}{dx} = \frac{6x+5y-7}{2x+18y-14}$

Ans: $(3y - 2x - 1)^2\ (x + 2y - 2) = 343c = c''$

3. $\frac{dy}{dx} + \frac{10x+8y-12}{7x+5y-9} = 0$

Ans: $(x + y - 1)^2\ (2x + y - 3)^3 = c$

4. (x – y – 2)dx + (x – 2y – 3)dy = 0

Sol. Given equation is $\frac{dy}{dx}=\frac{-x+y+2}{x-2y-3}$

$\Rightarrow a_1=-1, b_1=1, a_2=1, b_2=-2$

$\frac{a_1}{a_2}=\frac{-1}{1}, \frac{b_1}{b_2}=\frac{1}{-2} \quad \therefore \frac{a_1}{a_2}\neq\frac{b_1}{b_2}$

Let x = X + h, y = Y + k where $-h+k+2=0$ and $h-2k-3=0$ and $\frac{dy}{dx}=\frac{dY}{dX}$

Solving these equations h = 1, k = –1

Therefore $\frac{dY}{dX}=\frac{-X+Y}{X-2Y}$ which is homogeneous differential equation.

Put Y = VX so that $\frac{dY}{dX}=V+X\frac{dV}{dX}$

$$V+X\frac{dV}{dX}=\frac{X(-1+V)}{X(1-2V)}=\frac{-1+V}{1-2V}$$

$$X\frac{dV}{dX}=\frac{-1+V}{1-2V}-V=\frac{-1+V-V+2V^2}{1-2V}=\frac{2V^2-1}{1-2V}$$

$$\int\frac{(1-2V)dV}{2V^2-1}=\int\frac{dX}{X}$$

$$\int\frac{dX}{X}=\int\frac{-\frac{1}{2}(-4V)-1}{1-2V^2}dV$$

$$=\frac{1}{2}\int\frac{(-4VdV)}{1-2V^2}-\int\frac{dV}{1-2V^2}$$

$$\log|x|=-\frac{1}{2}\log|1-2V^2|-\frac{1}{2}\int\frac{dV}{\left(\frac{1}{\sqrt{2}}\right)^2-V^2}$$

$$=-\frac{1}{2}\log|1-2V^2|-\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\log\left|\frac{\frac{1}{\sqrt{2}}+V}{\frac{1}{\sqrt{2}}-V}\right|+\log c \quad 2\log|x|=-\log|1-2V^2|$$

$$-\frac{1}{2\left(\frac{1}{\sqrt{2}}\right)}\log\left|\frac{\frac{1}{\sqrt{2}}+V}{\frac{1}{\sqrt{2}}-V}\right|+\log C$$

$$2\log|x| + \log|1-2V^2| = -\frac{1}{\sqrt{2}}\log\left|\frac{1+V\sqrt{2}}{1-V\sqrt{2}}\right| + \log c$$

$$\log X^2(1-2V^2) = -\frac{1}{\sqrt{2}}\log\left|\frac{X+Y\sqrt{2}}{X-Y\sqrt{2}}\right| + \log c$$

$$\log|X^2-2Y^2| = \log c\left(\frac{X-Y\sqrt{2}}{X+Y\sqrt{2}}\right)^{1/\sqrt{2}}$$

$$\therefore X^2-2Y^2 = c\left(\frac{X-Y\sqrt{2}}{X+Y\sqrt{2}}\right)^{1/\sqrt{2}}$$

Substituting X = x–h = x – 1 , Y = y–k = y + 1

$$(x-1)^2-2(y+1)^2 = c\left(\frac{x-1-(y+1)\sqrt{2}}{x-1+(y+1)\sqrt{2}}\right)^{1/\sqrt{2}}$$

$$(x^2-2y^2-2x-4y-1) = c\left(\frac{x-y\sqrt{2}-1-\sqrt{2}}{x+y\sqrt{2}-1+\sqrt{2}}\right)^{1/\sqrt{2}}$$

5. (x – y)dy = (x + y + 1) dx

Ans: $\log c^2\left(x^2+y^2+x+y+\frac{1}{2}\right)$

6. (2x + 3y – 8)dx = (x + y – 3)dy

7. $\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$

8. $\frac{dy}{dx} = \frac{2x+9y-20}{6x+2y-10}$

Ans: c(x + 2y – 5) = (y – 2x)^2 = (2x – y)^2

Linear Equations:

A differential equation of the form $\frac{dy}{dx}+Py=Q$, where P and Q are functions of x only is called a linear differential equation of the first order in y.

Bernoulli's Equation

An equation of the form $\frac{dy}{dx}+Py=Qy^n$, where P and Q are functions of x only, is called a Bernoulli's equation.

EXERCISE – 11(E)

I. Find the I.F. of the following differential equations by transforming them into linear form.

1. $x\frac{dy}{dx}-y=2x^2\sec^2 2x$

Sol. $x\frac{dy}{dx}-y=2x^2\sec^2 2x$

$\frac{dx}{dy}-\frac{1}{x}y=2x\sec^2 2x$ which is linear in y .

$$\text{I.F.}=e^{\int p dx}=e^{\int -\frac{1}{x}}\log x=e^{-\log x}=e^{\log(1/x)}=\frac{1}{x}$$

2. $y\frac{dx}{dy}-x=2y^3$

Sol. $y\frac{dx}{dy}-x=2y^3 \Rightarrow \frac{dx}{dy}-\frac{1}{y}x=2y^2$ which is linear equation in x.

$$\text{I.F.}=e^{\int p dy}=e^{\int -\frac{1}{y}dy}=e^{-\log y}=e^{\log(1/y)}=\frac{1}{y}$$

II. Solve the following differential equations.

1. $\frac{dy}{dx}+y\tan x=\cos^3 x$

Sol. $\frac{dy}{dx}+y\tan x=\cos^3 x$ which is linear differential equation in y.

$$\text{I.F.}=e^{\int p\, dx}=e^{\int \tan x\, dx}=e^{\log(\sec x)}=\sec x$$

Solution of the equation is

y.I.F. = $y.\text{I.F}=\int Q.\ \text{I.F.}\ dx$

$\Rightarrow y\sec x=\int \sec x\cos^3 x\ dx=\int \cos^2 x\ dx$

$=\frac{1}{2}\int(1+\cos 2x)dx=\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+c$

$$\frac{2y}{\cos x} = x + \sin x \cdot \cos x + c$$

Solution is:

$$\mathbf{2y = x\cos x + \sin x \cdot \cos^2 x + c \cdot \cos x}$$

2. $\frac{dy}{dx} + y\sec x = \tan x$

Sol. $\frac{dy}{dx} + y\sec x = \tan x$ which is l.d.e. in y

$$\text{I.F.} = e^{\int \sec x dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

Sol is y.I.F. = $y.\text{I.F} = \int Q.\ \text{I.F.}\ dx$

$$y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)dx$$

$$= \int (\sec x \cdot \tan x + \tan^2 x)dx$$

$$= \int (\sec x \cdot \tan x + \sec^2 x - 1)dx$$

Solution is

$$y(\sec x + \tan x) = \sec x + \tan x - x + c$$

3. $\frac{dy}{dx} - y\tan x = e^x \sec x$ **which is l.d.e. in y.**

Sol. $\text{I.F.} = e^{-\int \tan x\, dx} = e^{\log \cos x} = \cos x$

Sol is y.I.F. = $y.\text{I.F} = \int Q.\ \text{I.F.}\ dx$

$$y\cos x = \int e^x \sec x \cos x dx = \int e^x dx = e^x + c$$

4. $x\frac{dy}{dx} + 2y = \log x$

Ans: $\frac{x^2}{2}\log x - \frac{x^2}{4} + c$

5. $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$

Sol. $\frac{dy}{dx} + \frac{1}{1+x^2} \cdot y = \frac{e^{\tan^{-1} x}}{1+x^2}$ which is a linear differential equation in y.

$$\text{I.F.} = e^{\int p dx} = e^{\int \frac{dx}{1+x^2}} = e^{\tan^{-1} x}$$

Sol **is** **y.I.F. =** $y.\text{I.F} = \int Q.\ \text{I.F.}\ dx$

$$y\cdot e^{\tan^{-1}x} = \int\frac{(e^{\tan^{-1}x})^2}{1+x^2}dx \quad \textbf{...(1)}$$

Consider $\int\frac{(e^{\tan^{-1}x})^2}{1+x^2}dx$ put $\tan^{-1}x = t \Rightarrow \frac{dx}{1+x^2} = dt$

$$= \int(e^t)^2 dt = \int e^{2t}dt = \frac{e^{2t}}{2} = \frac{e^{2\tan^{-1}x}}{2}$$

Solution is $y\cdot e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + \frac{c}{2}$

$$2y\cdot e^{\tan^{-1}x} = e^{2\tan^{-1}x} + c$$

6. $\frac{dy}{dx} + \frac{2y}{x} = 2x^2$

Sol. $\frac{dy}{dx} + \frac{2y}{x} = 2x^2$ which is l.d.e. in y.

$$\text{I.F.} = e^{\int\frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$$

Sol is y.I.F. = y.I.F = $\int$Q. I.F. dx

$$y\cdot x^2 = \int 2x^4 dx = \frac{2x^5}{5} + c$$

7. $\frac{dy}{dx} + \frac{4x}{1+x^2}y = \frac{1}{(1+x^2)^2}$

Ans: $y(1+x^2)^2 = \int dx = x + c$

8. $x\frac{dy}{dx} + y = (1+x)e^x$

Ans: $y\cdot x = x\cdot e^x + c$

9. $\frac{dy}{dx} + \frac{3x^2}{1+x^3}y = \frac{1+x^2}{1+x^3}$

Sol. $\frac{dy}{dx} + \frac{3x^2}{1+x^3}y = \frac{1+x^2}{1+x^3}$ which is linear differential equation in y.

$$\text{I.F.} = e^{\int\frac{3x^2}{1+x^3}dx} = e^{\log(1+x^3)} = 1+x^3$$

Sol is y.I.F. = y.I.F = $\int$Q. I.F. dx

$$y(1+x^3) = \int(1+x^2)dx = x + \frac{x^3}{3} + c$$

10. $\frac{dy}{dx} - y = -2e^{-x}$

Ans: $y = e^{-x} + ce^{x}$

11. $(1+x^2)\frac{dy}{dx} + y = \tan^{-1} x$ **.**

Sol. $\frac{dy}{dx} + \frac{1}{1+x^2} \cdot y = \frac{\tan^{-1} x}{1+x^2}$ which is linear differential equation in y.

I.F. $e^{\int\frac{dx}{1+x^2}} = e^{\tan^{-1} x}$

Sol is y.I.F. $=$ y.I.F $= \int Q.\ \text{I.F. } dx$

$$y \cdot e^{\tan^{-1} x} = \int \tan^{-1} x \frac{e^{\tan^{-1} x}}{1+x^2} dx$$

Put t = $\tan^{-1}$x so that $dt = \frac{dx}{1+x^2}$

R.H.S. $= \int t \cdot e^t dt = t \cdot e^t - \int e^t dt = t \cdot e^t - e^t$

Solution is : $y.e^{\tan^{-1} x} = e^{\tan^{-1} x}(\tan^{-1} x - 1) + c$

$$y = \tan^{-1} x - 1 + c \cdot e^{-\tan^{-1} x}$$

12. $\frac{dy}{dx} + y \tan x = \sin x$ **.**

Ans : $\int \tan x\, dx = \log \sec x + c$

III. Solve the following differential equations.

1. $\cos x \frac{dy}{dx} + y \sin x = \sec^2 x$

Sol. $\frac{dy}{dx} + \tan x \cdot y = \sec^3 x$ which is l.d.e in y

$$\text{I.F.} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

Sol is y.I.F. = y.I.F $= \int Q.\ \text{I.F. } dx$

$$y \cdot \sec x = \int \sec^4 x dx = \int (1+\tan^2 x)\sec^2 x dx$$

$$= \tan x + \frac{\tan^3 x}{3} + c$$

2. $\sec x \cdot dy = (y + \sin x)dx$

Sol. $\frac{dy}{dx} = \frac{y+\sin x}{\sec x} = y\cos x + \sin x \cdot \cos x$

$\frac{dy}{dx} - y\cos x = \sin x \cdot \cos x$ which is l..d.e in y

$\text{I.F.} = e^{-\int \cos x dx} = e^{-\sin x}$

Sol is $y.\text{I.F.} = y.\text{I.F} = \int Q.\ \text{I.F.}\ dx$

$$y \cdot e^{-\sin x} = \int e^{-\sin x} \cdot \sin x \cdot \cos x \cdot dx$$

Consider $\int e^{-\sin x} \cdot \sin x \cdot \cos x \cdot dx$

$t = -\sin x \Rightarrow dt = -\cos x\ dx$

$\int e^{-\sin x} \cdot \sin x \cdot \cos x\, dx = +\int e^t \cdot t\, dt$

$= t \cdot e^t - e^t + c = e^{-\sin x}(-\sin x - 1) + c$

$y \cdot e^{-\sin x} = -e^{-\sin x}(\sin x + 1) + c$

or $y = -(\sin x + 1) + c \cdot e^{\sin x}$

3. $x\log x \cdot \frac{dy}{dx} + y = 2\log x$

Ans: $y\log x = (\log x)^2 + c$

4. $(x + y + 1)\frac{dy}{dx} = 1$

Sol. $(x + y + 1)\frac{dy}{dx} = 1$

$\frac{dx}{dy} = x + y + 1 \Rightarrow \frac{dx}{dy} - x = y + 1$ which is l.d.e in x.

$\text{I.F.} = e^{\int p dy} = e^{\int -dy} = e^{-y}$

sol is $x.\text{I.F.} = \int Q.\text{IF}dy$

$$x \cdot e^{-y} = \int e^{-y}(y+1)dy = -(y+1)e^{-y} + \int e^{-y} dy$$

$$= -(y+1)e^{-y} - e^{-y} = -(y+2)e^{-y} + c$$

$$x = -(y+2) + c \cdot e^{y}$$

5. Solve $x(x-1)\dfrac{dy}{dx} - y = x^3(x-1)^3$**.**

Sol. $\dfrac{dy}{dx} - \dfrac{1}{x(x-1)}y = x^2(x-1)^2$ which is l.d.e in y

$$\text{I.F.} = e^{\int p dx} = e^{\int -\frac{dx}{x(x-1)}} = e^{\int\left(\frac{1}{x} - \frac{1}{x-1}\right)dx}$$

$$= e^{\log x - \log(x-1)} = e^{\log\frac{x}{x-1}} = \frac{x}{x-1}$$

Sol $\quad$ $y.\text{I.F.} = y.\text{I.F} = \int Q.\ \text{I.F.}\ dx$

$$y \cdot \frac{x}{x-1} = \int x^2(x-1)^2 \frac{x}{(x-1)} dx = \int x^3(x-1)dx$$

Hence solution is $\dfrac{xy}{x-1} = \dfrac{x^5}{5} - \dfrac{x^4}{4} + c$

6. $(x + 2y^3)\dfrac{dy}{dx} = y$

Ans: $x = y(y^2 + c)$

7. Solve $(1 - x^2)\dfrac{dy}{dx} + 2xy = x\sqrt{1 - x^2}$**.**

Ans $\quad$ $y = \sqrt{1 - x^2} + c(1 - x^2)$

8. $x(x-1)\dfrac{dy}{dx} - (x-2)y = x^3(2x-1)$

Ans: $y(x - 1) = x^2(x^2 - x + c)$

9. $\dfrac{dy}{dx}(x^2y^3 + xy) = 1$

Sol. $\dfrac{dy}{dx}(x^2y^3 + xy) = 1$

$$\frac{dx}{dy} = xy + x^2y^3$$

$$\Rightarrow \frac{dx}{dy} - xy = x^2y^3 \text{----}(1)$$

Which is Bernoulli's equation

Dividing with x^2,

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}y = y^3$$

Put $z = -\frac{1}{x}$ so that $\frac{dz}{dy} = \frac{1}{x^2}\frac{dx}{dy}$

$$\Rightarrow \frac{dz}{dy} + z \cdot y = y^3 \text{----}2)$$

which is linear differential equation in z

$$\text{I.F.} = e^{\int y dy} = e^{y^2/2}$$

Sol $z.\text{I.F} = \int Q.\ \text{I.F.}\ dy$

$$z \cdot e^{y^2/2} = \int y^3 e^{y^2/2} \cdot dy$$

put $\frac{y^2}{2} = t \Rightarrow y\,dy = dt$

$$= \int t \cdot dt \cdot e^t = e^t(t-1) = e^{y^2/2}\left(\frac{y^2}{2} - 1\right)$$

$$z \cdot e^{y^2/2} = e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + c$$

$$z = \frac{y^2}{2} - 1 + c \cdot e^{-y^2/2} \Rightarrow -\frac{1}{x} = \frac{y^2}{2} - 1 + c \cdot e^{-y^2/2}$$

$$-1 = x\left(\frac{y^2}{2} - 1 + c \cdot e^{-y^2/2}\right)$$

Hence solution is $1 + x\left(\frac{y^2}{2} - 1 + c \cdot e^{-y^2/2}\right) = 0$

10. $\frac{dy}{dx} + x \cdot \sin 2y = x^3 \cos^2 y$

$$\text{Ans} : \tan y = \frac{x^2 - 1}{2} + c \cdot e^{-x^2}$$

11. $y^2+\left(x-\frac{1}{y}\right)\frac{dy}{dx}=0$

Sol.

$$y^2+\left(x-\frac{1}{y}\right)\frac{dy}{dx}=0$$

$$\left(x-\frac{1}{y}\right)\frac{dy}{dx}=-y^2$$

$$\frac{dx}{dy}=\frac{x-1/y}{-y^2}=-\frac{x}{y^2}+\frac{1}{y^3}$$

$\frac{dx}{dy}+\frac{1}{y^2}\cdot x=\frac{1}{y^3}$ which is differential equation in x

$$\text{I.F.}=e^{\int\frac{1}{y^2}dy}=e^{-1/y}$$

Sol is $x.\text{I.F}=\int Q.\ \text{I.F.}\ dy$

$$x\cdot e^{-1/y}=\int\frac{e^{-1/y}}{y^3}dy \qquad \ldots(1)$$

$$\text{put} -\frac{1}{y}=z\Rightarrow\frac{1}{y^2}dy=dz$$

$$=\int z\cdot e^z dz=e^z(z-1)$$

$$x\cdot e^{-1/y}=-e^{-1/y}\left(-\frac{1}{y}-1\right)+c$$

$$\frac{x}{e^{1/y}}=\frac{1+y}{y\cdot e^{1/y}}+c$$

Hence solution is $xy = 1 + y + cy\, e^{1/y}$.

PROBLEMS FOR PRACTICE

1. **Find the order and degree of the differential equation** $\frac{d^2y}{dx^2}=-p^2y$.

Ans. Degree = 1, Order = 2

2. **Find the order and degree of** $\left(\frac{d^3y}{dx^3}\right)^2-3\left(\frac{dy}{dx}\right)^2-e^x=4$.

Ans. Degree = 2, Order = 3

3. $x^{1/2}\left(\frac{d^2y}{dx^2}\right)^{1/3} + x \cdot \frac{dy}{dx} + y = 0$ **has order 2, degree 1. Prove it.**

4. Find the order and degree of $\left(\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\right)^{6/5} = 6y$ **.**

Sol. Given equation is: $\left(\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\right)^{6/5} = 6y$

i.e. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = (6y)^{5/6}$

Order = 2, degree = 1

5. Find the order of the differential equations corresponding to y = c(x – c)^2 where c is an arbitrary constant.

Ans. Order = 1

6. Find the order of the differential equation corresponding to y = Ae^x + Be^{3x} + Ce^{5x} (A, B, C being parameters) is a solution.

Ans. Order = 3

7. Form the differential equation corresponding to y = cx – 2c^2, where c is a parameter.

Ans. $y = x\left(\frac{dy}{dx}\right) - 2\left(\frac{dy}{dx}\right)^2$.

8. Form the differential equation corresponding to y = Acos 3x + B sin 3x where A and B are parameters.

Ans. $\frac{d^2y}{dx^2} + 9y = 0$

9. Form the differential equation corresponding to the family of circles of radius r given by (x – a^2) + (y – b)^2 = r^2, where a and b are parameters.

Sol. We have : (x – a^2) + (y – b)^2 = r^2 …(1)

Differentiating (1) w.r.to x

$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$ …(2)

Differentiating (2) w.r.to x

$1 + (y-b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$ …(3)

From (2) $(x-a)=-(y-b)\frac{dy}{dx}$

Substituting in (1), we get

$$(y-b)^2\left(\frac{dy}{dx}\right)^2+(y-b)^2=r^2$$

$$(y-b)^2\left(\left(\frac{dy}{dx}\right)^2+1\right)=r^2 \qquad \ldots(4)$$

From (3) $(y-h)\frac{d^2y}{dx^2}=-\left(1+\left(\frac{dy}{dx}\right)^2\right)$

$$(y-h)=-\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)}{\left(\frac{d^2y}{dx^2}\right)}$$

Substituting in (4) : $\dfrac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^3}{\left(\frac{d^2y}{dx^2}\right)^2}=r^2$

i.e. $r^2\left(\frac{d^2y}{dx^2}\right)^2=\left(1+\left(\frac{dy}{dx}\right)^2\right)^3$

10. Form the differential equation corresponding to the family of circles passing through the origin and having centers on Y-axis.

Ans. $(x^2-y^2)\frac{dy}{dx}-2xy=0$

11. Express the following differential equations in the form f(x)dx + g(y)dy = 0.

i) $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$

ii) $y-x\frac{dy}{dx}=a\left(y^2+\frac{dy}{dx}\right)$

iii) $\frac{dy}{dx}=e^{x-y}+x^2\cdot e^{-y}=\frac{e^x}{e^y}+\frac{x^2}{e^y}$

iv) $\frac{dy}{dx} + x^2 = x^2 \cdot e^{3y}$

12. Find the general solution of $x + y\frac{dy}{dx} = 0$**.**

Ans. $x^2 + y^2 = 2c = c'$

13. Find the general solution of $\frac{dy}{dx} = e^{x+y}$**.**

Ans. $e^x + e^{-y} = c$

14. Solve $y^2 - x\frac{dy}{dx} = a\left(y + \frac{dy}{dx}\right)$

Ans. $y = \frac{c^a(y-a)}{a(x+a)^a}$

15. Solve $\frac{dy}{dx} = \frac{y^2 + 2y}{x - 1}$

Ans. $y = c^2(x-1)^2 (y+2)$

16. Solve $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

Ans. $y \sin y = x^2 \log x + c$

17. Find the equation of the curve, whose slope at any point is y/x^2 and which satisfies the condition y = 1 when x = 3.

Ans. $y = e^{\frac{x-3}{3x}}$

18. Solve y(1 + x)dx + x(1 + y)dy = 0

Ans. $x + y + \log(xy) = c$

19. Solve $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

Ans. $x = \log\left(1 + \tan\frac{x+y}{2}\right) + c$

20. Solve $(x-y)^2\frac{dy}{dx} = a^2$

Ans. $y = \frac{a}{2}\log\left|\frac{x-y-a}{x-y+a}\right| + c$

21. Solve $\sqrt{1+x^2}\sqrt{1+y^2}dx + xydy = 0$**.**

Ans. $\log|x| - \log(1+\sqrt{1+x^2}) + \sqrt{1+x^2} + \sqrt{1+y^2} = c$

22. Solve $\frac{dy}{dx} = \frac{x-2y+1}{2x-4y}$

Ans. $(x - 2y)^2 + 2x = c'$ where $c' = 2c$.

23. Solve $\frac{dy}{dx} = \sqrt{y-x}$

Ans. $x + c = 2\sqrt{y-x} + 2\log(\sqrt{y-x} - 1)$

24. Solve $\frac{dy}{dx} + 1 = e^{x+y}$

Ans. $x + e^{-(x+y)} + c = 0$

25. Solve $\frac{dy}{dx} = (3x+y+4)^2$

Ans. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{3x+y+4}{\sqrt{3}}\right) = x + c$

26. Solve $\frac{dy}{dx} - x\tan(y-x) = 1$

Ans. $\log|\sin(y-x)| = \frac{x^2}{2} + c$

27. Show that f(x, y) = 1 + $e^{x/y}$ **is a homogenous function of x and y.**

28. Show that $f(x,y) = x\sqrt{x^2+y^2} - y^2$ **is a homogenous function of x and y.**

29. Show that f(x, y) = x – y log y + y log x is a homogeneous function of x and y.

30. Express $(1+e^{x/y})dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$ **in the form** $\frac{dx}{dy} = F\left(\frac{x}{y}\right)$**.**

31. Express $(x\sqrt{x^2+y^2} - y^2)dx + xy\,dy = 0$ **in the form** $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$**.**

32. Express $\frac{dy}{dx} = \frac{y}{x+ye^{-2x/y}}$ **in the form** $\frac{dx}{dy} = F\left(\frac{x}{y}\right)$**.**

33. Solve $\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - xy}$.

Ans. $xy\sqrt{2xy - 3x^2} = c^3$

34. Solve $(x^2 + y^2)dx = 2xy\,dy$.

Ans. $cx(x^2 - y^2) = x$

35. Solve $xy^2 dy - (x^3 + y^3)dx = 0$.

Ans. $y^3 = 3x^3 \log cx$.

36. Solve $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$

Ans. $2x = (x - y)(\log x + c)$

37. Solve $x \sec\left(\frac{y}{x}\right)(ydx + xdy) = y \csc\left(\frac{y}{x}\right)(xdy - ydx)$.

Sol.

$$x \sec\left(\frac{y}{x}\right)(ydx + xdy) = y \csc\left(\frac{y}{x}\right)(xdy - ydx)$$

$$\Rightarrow x \sec\left(\frac{y}{x}\right)\left(y + x\frac{dy}{dx}\right) = y \csc\left(\frac{y}{x}\right)\left(x\frac{dy}{dx} - y\right)$$

$$x\frac{dy}{dx}\left(x \cdot \sec\left(\frac{y}{x}\right) - y \cdot \csc\left(\frac{y}{x}\right)\right) = -y\left[y \csc\left(\frac{y}{x}\right) + x \sec\left(\frac{y}{x}\right)\right]$$

$$\frac{dy}{dx} = \frac{-y\left(y \csc\left(\frac{y}{x}\right) + x \sec\left(\frac{y}{x}\right)\right)}{x\left(x \sec\left(\frac{y}{x}\right) - y \csc\left(\frac{y}{x}\right)\right)}$$

This is a homogeneous equation.

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v\left(\frac{v \csc v + \sec v}{v \csc v - \sec v}\right)$$

$$= \frac{v\left(\frac{v}{\sin v} + \frac{1}{\cos v}\right)}{\left(v\frac{1}{\sin v} - \frac{1}{\cos v}\right)} = \frac{v(v \cos v + \sin v)}{v \cos v - \sin v}$$

$$x\frac{dv}{dx} = \frac{v(v\cos v + \sin v)}{v\cos v - \sin v} - v$$

$$= \frac{v(v\cos v + \sin v - v\cos v + \sin v)}{v\cos v - \sin v}$$

$$= \frac{2v\sin v}{v\cos v - \sin v}$$

$$\int \frac{v\cos v - \sin v}{v\sin v}dv = 2\int \frac{dx}{x}$$

$$\int \frac{\cos v}{\sin v}dv - \int \frac{1}{v}dv = 2\int \frac{dx}{x}$$

$$\log \sin v - \log v = 2\log x + \log c$$

$$\log\left(\frac{\sin v}{v}\right) = \log cx^2 \Rightarrow \frac{\sin v}{v} = cx^2$$

$$\frac{x}{y}\sin\left(\frac{y}{x}\right) = cx^2 \Rightarrow \sin\left(\frac{y}{x}\right) = cxy$$

38. Give the solution of $x\sin^2\frac{y}{x}dx = ydx - xdy$ **which passes through the point (1, π/4).**

Ans. $\cot\frac{y}{x} = \log x + 1$

39. Solve $(x^3 - 3xy^2)dx + (3x^2y - y^3)dy = 0$.

Ans. $\dfrac{\sqrt{\dfrac{y^2}{x^2} - 1}}{\dfrac{y^2}{x^2} + 1} = cx$

40. Transform the following two differential equations into linear form.

Ans. $\frac{dy}{dx} + Py = Q$

41. $(x + 2y^3)\frac{dy}{dx} = y$

Ans. $\frac{dx}{dy} + Px = Q$

Find I.F. of the following two differential equations by transforming them into linear form.

42. $(\cos x)\frac{dy}{dx} + y\sin x = \tan x$

Ans. $\text{I.F.} = e^{\int \log \sec x} = \sec x$

43. Solve $\left(2x-10y^3\right)\frac{dy}{dx}+y=0$

Ans. $I.F.=e^{\int\frac{2}{y}dy}=e^{2\log y}=e^{\log y^2}=y^2$

44. Solve $\left(1+x^2\right)\frac{dy}{dx}+2xy-4x^2=0$

Ans. $y(1+x^2)=\int 4x^2\,dx=\frac{4x^3}{3}+c$

45. Solve $\frac{1}{x}\cdot\frac{dy}{dx}+y\cdot e^x=e^{(1-x)e^2}$.

Ans. $2y\cdot e^{(x-1)e^x}=x^2+2c$

46. Solve $\sin^2 x\cdot\frac{dy}{dx}+y=\cot x$.

Ans. $y\cdot e^{-\cot x}=(\cot x+1)e^{-\cot x}+c$

47. Find the equation of the equation $x(x-2)\frac{dy}{dx}-2(x-1)y=x^3(x-2)$ **which satisfies the condition that y = 9 when x = 3.**

Ans. $\frac{y}{x(x-2)}=x+2\log(x-2)$

48. Solve $(1+y^2)dx=(\tan^{-1}y-x)dy$.

Ans. $x\cdot e^{\tan^{-1}y}=e^{\tan^{-1}y}(\tan^{-1}y-1)+c$.