CHAPTER 10

NUMERICAL INTEGRATION

Topics:

1. TRAPEZOIDAL RULE

2.SIMPSON'S RULE

NUMERICAL INTEGRATION

TRAPEZOIDAL RULE

Let y = f(x) be a continuous function defined on [a, b]. Divide the interval on x-axis into n equal parts each of length $h = \frac{b-a}{n}$. Let a = x_0, x_1, x_2, …….., x_n = b be the abscissae of successive points of division and y_r = f (x_r) for r = 0, 1, 2 …….n then

$$\int_a^b ydx = \frac{h}{2}\left(y_0 + 2(y_1 + y_2 + \ldots\ldots + y_{n-1}) + y_n\right)$$

SIMPSON'S RULE

Let y = f (x) be a continuous function defined on [a, b]. Divide the interval on x-axis into n = 2m (even integer) equal parts each of length $h = \frac{b-a}{n} = \frac{b-a}{2m}$ .Let a = x_0, x_1, x_2, ……., x_n = b be the abscissae of the successive points of division and y_r = f (x_1) for r = 0, 1, 2, …. n then

$$\int_a^b ydx = \frac{h}{3}\left(y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 - + \ldots.)\right)$$

It is also known as Simpson's $\frac{1}{3}$ rule.

EXERCISE -- 10(B)

1. Given that e = 2.72 , $e^2 = 7.39,\ e^3 = 20.09,\ e^4 = 54.60,$ **evaluate** $\int_0^4 e^x\, dx$ **approximately using Simpson's rule.**

Sol: Given a = 0, b=4 and $h = \frac{b-a}{2n} = \frac{4-0}{4} = 1$

Let the curve be y = e^x

X	0	1	2	3	4
y= e^x	1	2.72	7.39	20.09	54.60

Simpson's rule

$$\int_a^b y dx = \frac{h}{3}\left(y_0 + y_n + 4\left(y_1 + y_3 + \ldots\right) + 2\left(y_2 + y_4 - + \ldots.\right)\right)$$ where n is even.

$$\Rightarrow \int_0^4 e^x\,dx = \frac{h}{3}\left[\left(y_0 + y_{2n}\right) + 4\left(y_1 + y_3 ...... + y_{2n-1}\right) + 2\left(y_2 + y_4 + ... + y_{2n-2}\right)\right]$$

$$= \frac{1}{3}\left[\left(1+54.60\right)+4\left(2.72+20.09\right)+2\left(739\right)\right]$$

$$= \frac{1}{3}\left[55.60+4\left(22.81\right)+14.78\right]$$

$$= \frac{1}{3}\left(70.38+91.24\right) = \frac{161.62}{3} = 53.8733$$

2. A river is 80 meters width. The depth d-(in meters) of the river at a distance x from the bank is given by the following table

x	**0**	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**
d	**0**	**4**	**7**	**9**	**12**	**15**	**17**	**8**	**0**

Find approximately the area of the cross section of the river using Simpson's rule.

Sol: Given h = 10, a=0 and b=80

Simpson's $\int_a^b y dx = \frac{h}{3}\left(y_0 + y_n + 4\left(y_1 + y_3 + \ldots\right) + 2\left(y_2 + y_4 - + \ldots.\right)\right)$

Area of cross section = $\int_0^{80} y\,dx = \frac{10}{3}\left[\left(0+0\right)+4\left(4+9+15+8\right)+2\left(7+12+17\right)\right]$

$$= \frac{10}{3}\left(144+72\right) = \frac{10}{3}\left(216\right) = 720 \text{ sq.units}$$

3. Find approximately the value of π from $\int_0^1 \frac{dx}{1+x^2}$ using Simpson's rule by dividing (0,1)into 4 equal parts.

Sol: Given n = 4, h = $\frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$, $y = \frac{1}{1+x^2}$

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$y = \frac{1}{1+x^2}$	1	$\frac{16}{17}$	$\frac{4}{5}$	$\frac{16}{25}$	$\frac{1}{2}$

SIMPSONS RULE

$$\int_a^b y dx = \frac{h}{3}\left(y_0 + y_n + 4\left(y_1 + y_3 + \ldots\right) + 2\left(y_2 + y_4 - + \ldots.\right)\right), \text{ n is even}$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\left(y_0 + y_n + 4\left(y_1 + y_3 + \ldots\right) + 2\left(y_2 + y_4 - + \ldots.\right)\right)$$

$$= \frac{1}{12}\left[\left(1+\frac{1}{2}\right)+4\left(\frac{16}{17}+\frac{16}{25}\right)+2\left(\frac{4}{5}\right)\right]$$

$$= \frac{1}{12}\left[1.5+4\left(0.9412+0.64\right)+1.6\right]$$

$$= \frac{1}{12}\left(3.1+6.3248\right) = \frac{9.4248}{12} = 0.7854$$

But $\int_0^1 \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$

$\therefore \frac{\pi}{4} = 0.7854 \Rightarrow \pi = 3.1416$.

4. Calculate the approximate value of $\int_0^1 (1+x^2)\,dx$ by dividing (0,1) into 5 equal parts using Trapezoidal rule.

Sol: given a =0 and b= 1

$h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$, equation of the curve is $y = 1+x^2$

x	0	0.2	0.4	0.6	0.8	1.0
$y = 1+x^2$	1	1.04	1.16	1.36	1.64	2

Trapezoidal rule

$$\int_a^b f(x)dx \cong \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

$$\int_0^1 (1+x^2)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 \dots + y_{n-1})\right]$$

$= \frac{0.2}{2}\left[(1+2)+2(1.04+1.16+1.36+1.64)\right]$ = 0.1 (3 + 2(5.2)) = (0.1) (3 + 10.4)

= (0.1) (13 .4) = 1.34

5. Calculate the approximate value of the following integrals using Simpson's rule $\int_1^3 \frac{1}{x}dx$, by taking n = 4.

Sol: given a = 1 , b = 3 , n = 4

$h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$ and the curve is $y = \frac{1}{x}$

x	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$y = \frac{1}{x}$	1	0.666	0.5	0.4	0.333

Simpson’s rule

$$\int_a^b ydx = \frac{h}{3}\left(y_0 + y_n + 4\left(y_1 + y_3 + \ldots\right) + 2\left(y_2 + y_4 - + \ldots.\right)\right)$$, n is even

$$\int_1^3 \frac{1}{x}\,dx = \frac{h}{3}\left(y_0 + y_n + 4\left(y_1 + y_3 + \ldots\right) + 2\left(y_2 + y_4 - + \ldots.\right)\right)$$

$$= \frac{1}{6}\left[\left(1+0.333\right)+4\left(0.666+0.4\right)+2\left(0.5\right)\right]$$

$$= \frac{1}{6}\left(1.333+4.264+1\right) = \frac{6.597}{6} = 1.0995$$

By Simpson’s rule $\int_1^3 \frac{1}{x}dx$

$$= \frac{1}{12}\left[\left(1+0.33\right)+4\left(2.17\right)+2\left(1.57\right)\right]$$

$$= \frac{1}{12}\left(1.33+8.68+3.14\right) = \frac{13.15}{12}$$

= 1.1 (approximately)

II.

1. Use Simpson’s rule to evaluate $\int_1^7 \frac{1}{x}\,dx$ **approximately by taking n = 6 and hence find the approximate value of** $\log_e^{7}$ **three places of decimals.**

Sol: Given a=1 ,b =7 . then $h = \frac{b-a}{n} = \frac{7-1}{6} = 1$ and given curve is $y = \frac{1}{x}$

x	1	2	3	4	5	6	7
$y = \frac{1}{x}$	1	0.5	0.333	0.25	0.2	0.166	0.143

Simpson’s rule

$$\int_a^b ydx = \frac{h}{3}\left(y_0 + y_n + 4\left(y_1 + y_3 + \ldots\right) + 2\left(y_2 + y_4 - + \ldots.\right)\right)$$

$$\int_1^7 \frac{1}{x}\,dx = \frac{h}{3}\left(y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 - + \ldots.)\right)$$

$$= \frac{1}{6}[(1+0.143)+4(0.667+0.4+0.286)+0.222+0.182.0.154)+2(0.5+0.333+0.25+0.2+0.167)]$$

$$= \frac{1}{6}[1.143+7.652+2.9] = \frac{11.695}{6} = 1.949$$

2. Calculate the approximate value of $\int_0^6 \sqrt{1+x^2}\,dx$, **by taking n = 6 in Simpson's rule.**

Sol: Given n = 6,a=0 , b=6 then $h = \frac{b-a}{n} = \frac{6-0}{6} = 1$ and $y = \sqrt{1+x^2}$

x	0	1	2	3	4	5	6
y	1	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{10}$	$\sqrt{17}$	$\sqrt{26}$	$\sqrt{37}$

Simpson's rule

$$\int_a^b y\,dx = \frac{h}{3}\left(y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 - + \ldots.)\right)$$

$$\int_0^6 \sqrt{1+x^2}\,dx = \frac{1}{3}\left[\left(1+\sqrt{37}\right)+4\left(\sqrt{2}+\sqrt{10}\right)+\sqrt{26}+2\left(\sqrt{5}+\sqrt{17}\right)\right]$$

$$= \frac{1}{3}\left(1+6.08+4(1.41+3.16+5.1)+2(2.24+4.12)\right)$$

$$= \frac{1}{3}[7.08+38.68+12.72] = \frac{57.48}{3} = 19.16$$

3. Evaluate $\int_1^2 x^2\,dx$ **approximately by taking n = 4 , in (i) Trapezoidal rule (ii) Simpson's rule .**

Sol: a=1, b=2 and n=4

Therefore $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$

Given curve is $y = x^2$

x	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2
y	1	$\frac{25}{16}$	$\frac{9}{4}$	$\frac{49}{16}$	4

i) Trapezoidal rule:

$$\int_a^b f(x)dx \cong \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})]$$

$$\int_1^2 x^2 dx = \frac{1}{8}\left[(1+4)+2\left(\frac{25}{16}+\frac{9}{4}+\frac{49}{16}\right)\right] = \frac{1}{8}\left[5+2\left(\frac{74}{16}+\frac{9}{4}\right)\right]$$

$$= \frac{1}{8}\left[5+2\left(\frac{74+36}{16}\right)\right] = \frac{1}{8}\left(5+\frac{55}{4}\right) = \frac{1}{8}\left(\frac{20+55}{4}\right) = \frac{75}{32} = 2.343$$

ii) Simpson's rule:

$$\int_a^b ydx = \frac{h}{3}\left(y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 - + \ldots.)\right)$$

$$\int_1^2 x^2 dx = \frac{1}{12}\left[(1+4)+4\left(\frac{25}{16}+\frac{49}{16}\right)+2.\frac{9}{4}\right]$$

$$= \frac{1}{12}\left[5+4.\frac{74}{16}+\frac{9}{2}\right] = \frac{1}{12}(5+23) = \frac{28}{12} = 2.333.$$

4. Calculate the approximate value of $\int_{-3}^{3} x^4 dx$ by using (i) Trapezoidal rule (ii) Simpson's rule by dividing [-3,3] into 6 equal parts.

Sol: a= -3 , b=3 and n=6

Therefore $h = \frac{b-a}{n} = \frac{3-(-3)}{6} = \frac{6}{6} = 1$ and given curve is $y = x^4$

x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81

i) Trapezoidal rule

$$\int_a^b f(x)dx \cong \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})]$$

$$\int_{-3}^{3} x^4\, dx = \frac{1}{2}\left[(81+81) + 2(16+1+0+1+16)\right] = \frac{1}{2}(162+68) = \frac{230}{2} = 115$$

ii) Simpson's rule : $\int_a^b y dx = \frac{h}{3}\left(y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 - + \ldots.)\right)$

$$\int_{-3}^{3} x^4 dx = \frac{1}{3}\left[(81+81) + 4(16+0+16) + 2(1+1)\right]$$

$$= \frac{1}{3}[162+128+4] = \frac{294}{3} = 98$$

PROBLEMS FOR PRACTICE

1 . Dividing [0,6] into 6 equal parts, evaluate $\int_0^6 x^3\, dx$ **approximately, by using (i) Trapezoidal rule (ii) Simpson's rule** .

Sol: Given a = 0 , b = 6 , n = 6 then $h = \frac{b-a}{n} = \frac{6-0}{6} = 1$

Functions is $y = x^3$

x	0	1	2	3	4	5	6
y	0	1	8	27	64	125	216

i) Trapezoidal rule

$$\int_a^b f(x)dx \cong \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})]$$

$$\int_0^6 x^3\ dx = \frac{1}{2}\left[(0+216)+2(1+8+27)+64+125\right]$$

$$=\frac{1}{2}\left[216+2(225)\right]=\frac{1}{2}(216+450)=\frac{666}{2}=333$$

ii) Simpson's rule $\int_a^b y dx = \frac{h}{3}\left(y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 - + \ldots.)\right)$

$$\int_0^6 x^3 dx = \frac{1}{3}\left[(0+216)+4(1+27+125)+2(8+64)\right]$$

$$= \frac{1}{3}\left[(216+4(153)+2(72))\right]$$

$$= \frac{1}{3}(216+612+144)=\frac{972}{3}=324$$

2.Evaluate $\int_0^1 \sqrt{1+x^3}\ dx$ **approximately by dividing (0,1) into 10 subintervals of same length by i) Trapezoidal rule, ii) Simpson's rule**

Sol: Given a = 0, b = 1, n = 10

Then $h=\frac{b-a}{n}=\frac{1-0}{10}=0.1$ and the function is $y=\sqrt{1+x^3}$

X	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Y	1	1.0004	1.004	1.013	1.031	1.060	1.103	1.159	1.230	1.318	1.414

i) Trapezoidal rule:

$$\int_a^b f(x)dx \cong \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+\ldots.+y_{n-1})]$$

$$\int_0^1 \sqrt{1+x^3}\ dx = \frac{0.1}{2}\left[(1+1.41)+2(1.0004+1.004+1.013+1.031+1.060+1.103+1.159+1.230+1.315)\right]$$

$$= \frac{0.1}{2}\left[2.414+2(9.915)\right]=\frac{22.244}{20}=1.1122$$

ii) Simpson's rule: $\int_a^b y dx = \frac{h}{3}\left(y_0 + y_n + 4\left(y_1 + y_3 + \ldots\right) + 2\left(y_2 + y_4 - + \ldots.\right)\right)$

$$\int_0^1 \sqrt{1+x^3}\, dx = \frac{0.1}{3}\left[(1+1.414)+4(1.0004+1.013+1.060+1.159+1.315)+2(1.004+1.031+1.103+1.230)\right]$$

$$= \frac{0.1}{3}\left[2.414+4(5.5474)+2(4.368)\right]$$

$$= \frac{0.1}{3}\left(2.414+22.1896+8.736\right) = \frac{33.3396}{30}$$

$= 1.11132.$

3. The velocity of a train which starts from rest is given by the following table, the time being recorded in minutes from the start and speed in kms.

Minute	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
Kmph	**10**	**18**	**25**	**29**	**32**	**20**	**11**	**5**	**2**	**0**

Estimate approximately the total distance run in 20 minutes by i) Simpson's rule ii) Trapezoidal rules.

Sol: Since the train starts from rest , therefore time t = 0 then its velocity v = 0. Therefore

Time	0	2	4	6	8	10	12	14	16	18	20
Kmph	0	10	18	25	29	32	20	11	5	2	0

Let s be the distance traveled in time t with velocity υ.

$$\frac{ds}{dt} = \upsilon \Rightarrow s = \int \upsilon\, dt$$

h = 20 minutes = $\frac{1}{30}$ hrs .

i) Simpson's rule:

$$\int_0^{20} \upsilon\, dt = \frac{h}{3}\left[\left(y_0 + y_{10}\right) + 4\left(y_1 + y_3 + y_5 + y_7 + y_9\right) + 2\left(y_2 + y_4 + y_6 + y_8\right)\right]$$

$$= \frac{1}{90}\left[(0+0)+4(10+25+32+11+2)+2(18+29+20+5)\right]$$

$$= \frac{1}{90}(320+144) = \frac{464}{90} = 5.16 \text{ kms}$$

ii) Trapezoidal rule :

$$\int_0^{20} \upsilon \, dt = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2).....+ y_9\right]$$

$$= \frac{1}{60}\left[(0+0)+2(10+18+25+29+32+20+11+5+2)\right]$$

$$= \frac{304}{60} = 5.07 \text{ kms} .$$