CHAPTER 9

DEFINITE INTEGRATION

TOPICS:

1.The definite integral

2.Interpretation of definite integral

3.Fundamental theorem of integral calculus

4.Properties of definite integral

5. Reduction formulae

DEFINITE INTEGRAL

Let f(x) be a function defined on [a, b]. If $\int f(x)\,dx = F(x)$, then F(b) – F(a) is called the definite integral of f(x) over [a, b]. It is denoted by $\int_a^b f(x)dx$. The real number ‘a’ is called the lower limit and the real number ‘b’ is called the upper limit.
This is known as fundamental theorem of integral calculus.

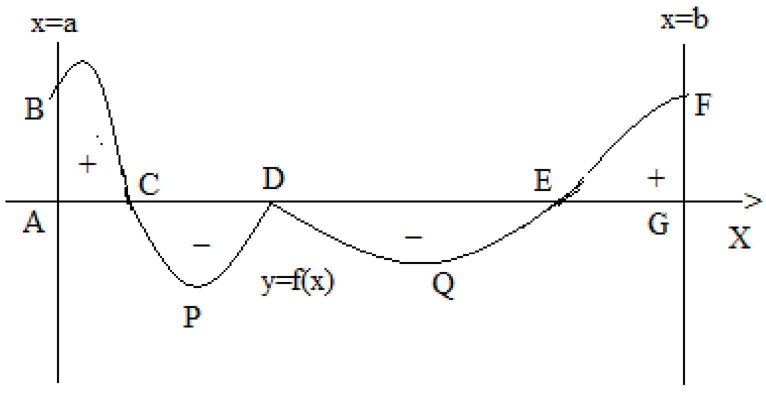
THEOREM

$\int_a^b f(x)dx = \int_a^b f(t)dt$ **i.e., definite integral is independent of its variable.**

Geometrical interpretation of definite integral

If f(x)>0 for all x in [a, b] then $\int_a^b f(x)\,dx$ **is numerically equal to the area bounded by the curve y =f(x), the x-axis and the lines x=a and x=b i.e.,** $\int_a^b f(x)\,dx$ **.**

In general, $\int_a^b f(x)\,dx$ **represents to algebraic sum of the areas of the figures bounded by the curve y =f(x) , the x axis and the lines x=a and x=b.. the areas above x-axis are taken with plus sign and the areas below x-axis are taken with minus sign i.e.,**



$\int_a^b f(x)\,dx =$ **area ABC – area CPD –area EQE + area EFG.**

PROPERTIES OF DEFINITE INTEGRALS

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ **.**

Proof : let $\int f(x)dx = F(x)$ **then**

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) = -[F(a) - F(b)] = -[F(x)]_b^a = -\int_b^a f(x)dx$$

2 . If a < c < b then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ **.**

Proof : Let $\int f(x)dx = F(x)$ **.**

Then $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

R.H.S= $\int_a^c f(x)dx + \int_c^b f(x)dx = [F(x)]_a^c + [F(x)]_c^b$

$$= F(c) - F(a) + F(b) - F(c) = F(b) - F(a) = \int_a^b f(x)dx = \text{L.H.S}$$

3 . $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof : Put a – x = t. $\Rightarrow$ **dx = –dt**

L.L, x = 0 $\Rightarrow$ **t = a**

U.L. x=a$\Rightarrow$ **t=0.**

$$\text{R.HS.} = \int_0^a f(a-x)dx = \int_a^0 f(t)(-dt) = -\int_a^0 f(t)dt = \int_0^a f(t)dt = \int_0^a f(x)dx = \text{L.H.S.}$$

THEOREM

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Proof : Put a + b – x = t, then –dx = dt $\Rightarrow$ **dx = –dt**

L.L, x = a $\Rightarrow$ **t = b**

U.L. x=b$\Rightarrow$ **t=a.**

$$\text{R.HS.} = \int_a^b f(a+b-x)dx = \int_b^a f(t)(-dt) = -\int_b^a f(t)dt = \int_a^b f(t)dt = \int_a^b f(x)dx = \text{LHS}$$

THEOREM

$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$**, if f(x) is an even function= 0, if f(x) is an odd function.**

Proof : Since –a < 0 < a, $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$

In the 1st integral of RHS,

Put x = –t, then dx = –dt,

L.L, x = -a ⇒ t = a

U.L. x=0⇒ t=0.

$$\therefore \int_{-a}^{a} f(x)dx = \int_{a}^{0} f(-t)(-dt) + \int_{0}^{a} f(x)dx = -\int_{a}^{0} f(-t)dt + \int_{0}^{a} f(x)dx$$

$$= \int_{0}^{a} f(-t)dt + \int_{0}^{a} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx ------(1)$$

Case I : If f(x) is an even function then f(–x) = f(x)

Then from (1),

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

Case II :If f(x) is an odd function then f(–x) = –f(x)

From (1),

$$\therefore \int_{-a}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0$$

THEOREM

$\int_{0}^{2a} f(x)dx = 2\int_{0}^{a} f(x)dx$ **if f(2a – x) = f(x)s= 0 if f(2a – x) = –f(x).**

Proof :

Since 0 < a < 2a, $\int_{0}^{2a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{a}^{2a} f(x)dx$ **---(1)**

In the 2nd integral of rhs, Put 2a – t = x, then –dx = dt ⇒ dx = –dt

L.L, x = a ⇒ t = a
U.L. x=2a⇒ t=0.

$$\int_a^{2a} f(x)dx = \int_a^0 f(2a-t)(-dt) = -\int_a^0 f(2a-t)dt = \int_0^a f(2a-t)dt = \int_0^a f(2a-x)dx$$

From (1),

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx \text{ ----(2)}$$

Case I : if f(2a – x) = f(x)
From (2),

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$$

Case II : if f(2a – x) = –f(x)
From (2),

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx = 0$$

THEOREM :

If f(x) is a periodic function with period 'T' then $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$ **.**

Proof : Let S(n) be the statement that $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$ **for n€N.**

Let n=1, then

$$\int_0^{1T} f(x)dx = \int_0^T f(x)dx = 1\int_0^T f(x)dx$$

∴ S(1) is true.
Assume that S(k) is true.

$$\therefore \int_0^{kT} f(x)dx = k\int_0^T f(x)dx$$

Now $\int_0^{(k+1)T} f(x)dx = \int_0^{kT} f(x)dx + \int_{kT}^{(k+1)T} f(x)dx = k\int_0^T f(x)dx + \int_{kT}^{(k+1)T} f(x)dx$

In the 2nd integral of rhs

Put x = kT + t, then dx = dt.

x = kT, (k + 1)T ⇒ t = 0, t = a.

$$\int_{kT}^{(k+1)T} f(x)dx = \int_0^T f(ka+t)dt = \int_0^T f(t)dt = \int_0^T f(x)dx$$

[∵ f(x) is a periodic function with period 'T']

$$\therefore \int_0^{(k+1)T} f(x)dx = k\int_0^T f(x)dx + \int_0^{(k+1)T} f(x)dx$$

$$= k\int_0^T f(x)dx + \int_0^T f(x)dx = (k+1)\int_0^T f(x)dx$$

∴ S(k + 1) is true. By principle of Mathematical Induction S(n) is true, ∀ n ∈ N.

$$\therefore \int_0^{nT} f(x)dx = n\int_0^T f(x)dx.$$

THEOREM

If f(x) is an integrable function on [a, b] and g(x) is derivable on [a, b] then

$$\int_a^b (fog)(x)g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx.$$

EXERCISE – 9(A)

I. Evaluate the following definite integrals.

1. $\int_0^a (a^2x - x^3)dx$

Sol. $\int_0^a (a^2x - x^3)dx = \left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{a^4}{2} - \frac{a^4}{4} = \frac{a^4}{4}$

2. $\int_2^3 \frac{2xdx}{1+x^2}$

Sol. $\int_2^3 \frac{2xdx}{1+x^2} = \left[\ln|1+x^2|\right]_2^3 = \ln 10 - \ln 5 = \ln(10/5) = \ln 2$

3. $\int_0^{\pi}\sqrt{2+2\cos\theta}\,d\theta$

Sol. $\int_0^{\pi}\sqrt{2+2\cos\theta}\,d\theta=\int_0^{\pi}\sqrt{2}\cdot\sqrt{2}\sqrt{\cos^2\frac{\theta}{2}}\,d\theta=\int_0^{\pi}2\cos(\theta/2)\,d\theta$

$=\left[4\sin\frac{\theta}{2}\right]_0^{\pi}=4\left(\sin\frac{\pi}{2}-\sin 0\right)=4$

4. $\int_0^{\pi}\sin^3 x\cdot\cos^3 x\,dx$

Sol. $\int_0^{\pi}\sin^3 x\cdot\cos^3 x\,dx=\int_0^{\pi}\sin^3(\pi-x)\cos^3(\pi-x)dx$

$=-\int_0^{\pi}\sin^3 x\cos^3 x\;dx=-I$

$\Rightarrow 2I=0\Rightarrow I=0=-I$

$2I=0\Rightarrow I=0$

5. $\int_0^{2}|1-x|\,dx$

Sol. $\int_0^{2}|1-x|\,dx=\int_0^{1}-(x-1)dx+\int_1^{2}(x-1)dx$

$=\int_0^{1}(-x+1)dx+\int_1^{2}(x-1)dx$

$=\left[\frac{-x^2}{2}+x\right]_0^1+\left[\frac{x^2}{2}-x\right]_1^2$

$=-\frac{1}{2}+1+\left(\frac{4}{2}-2\right)-\left(\frac{1}{2}-1\right)=\frac{1}{2}+\frac{1}{2}=1$

6. $\int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^x}dx$

Sol. Let $I=\int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^x}dx$ **…(i)**

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos(\pi/2 - \pi/2 - x)dx}{1 + e^{-x}} \left(\because \int_a^b f(x)dx = \int_a^b f(a + b - x)dx \right)$$

$$= \int_{-\pi/2}^{\pi/2} \frac{e^x \cos x dx}{1 + e^x} ----(2)$$

Adding (1) and (2) ,

$$2I = \int_{-\pi/2}^{\pi/2} \frac{\cos x(1 + e^x)}{1 + e^x} dx = \int_{-\pi/2}^{\pi/2} \cos x dx$$

$$2I = 2\int_0^{\pi/2} \cos x \, dx \, (\because \cos x \text{ is even function})$$

$$\Rightarrow I = [\sin x]_0^{\pi/2} \Rightarrow I = 1$$

7. $\int_0^1 \frac{dx}{\sqrt{3 - 2x}}$

Sol.

$$\int_0^1 \frac{dx}{\sqrt{3 - 2x}} = \left(\frac{2\sqrt{3 - 2x}}{-2} \right)_o^1 = -\left(\sqrt{3 - 2.1} - \sqrt{3 - 2.0} \right) = -\left(1 - \sqrt{3}\right) = \left(\sqrt{3} - 1\right)$$

8. $\int_0^a (\sqrt{a} - \sqrt{x})^2 dx$

Sol. $\int_0^a (\sqrt{a} - \sqrt{x})^2 dx = \int_0^a (a + x + 2\sqrt{a}\sqrt{x})dx$

$$= \left[ax + \frac{x^2}{2} - 2\sqrt{a} \cdot x^{3/2} \frac{2}{3} \right]_0^a$$

$$a^2 + \frac{a^2}{2} - \frac{4}{3}a^2 = \frac{6a^2 + 3a^2 - 8a^2}{6} = \frac{1}{6}a^2$$

9. $\int_0^{\pi/4} \sec^4 \theta d\theta$

$$\int_0^{\pi/4} \sec^4 \theta d\theta = \int_0^{\pi/4} \sec^2 \theta.\sec^2 \theta d\theta = \int_0^{\pi/4} \sec^2 \theta(1+\tan^2 \theta)d\theta$$

Sol. Let $= \int_0^{\pi/4} \left(\sec^2 \theta + \sec^2 \theta \tan^2 \theta\right)d\theta = \int_0^{\pi/4} \sec^2 \theta d\theta + \int_0^{\pi/4} \tan^2 \theta \sec^2 \theta d\theta$

$$= \tan\theta)_o^{\pi/4} + \left(\frac{\tan^3 \theta}{3}\right)_o^{\pi/4} = 1 - 0 + \frac{1}{3}(1-0) = \frac{4}{3}$$

10. $I = \int_0^3 \frac{x}{\sqrt{x^2+16}} dx$

Asn: 1

11. $\int_0^1 x \cdot e^{-x^2} dx$

Sol. $\int_0^1 x \cdot e^{-x^2} dx = \frac{1}{2}\int_0^1 2xe^{-x^2} dx, \quad \text{put} - x^2 = t$

$\Rightarrow -2xdx = dt \Rightarrow 2xdx = -dt$

$x = 1 \Rightarrow t = 1, \ x = 0 \Rightarrow t = 0$

$I = \frac{1}{2}\int_0^{-1} -e^t dt = \frac{1}{2}\left[-e^t\right]_0^{-1}$

$= \frac{1}{2}\left[e^0 - e^{-1}\right] = \frac{1}{2}\left(1 - \frac{1}{e}\right)$

12. $I = \int_1^5 \frac{dx}{\sqrt{2x-1}}$

Ans:2

II. Evaluate the following integrals.

1. $\int_0^4 \frac{x^2}{1+x} dx$

Sol. $\int_0^4 \frac{x^2}{1+x} dx = \int_0^4 \frac{x^2 - 1 + 1}{1+x} dx \Rightarrow I = \int_0^4 (x-1)dx + \int_0^4 \frac{dx}{1+x}$

$$=\left[\frac{x^2}{2}-x\right]_0^4+\left[\log(1+x)\right]_0^4$$

$$=\frac{4^2}{2}-4+\log 5-\log 1=4+\log 5$$

2. $\int_{-1}^{2}\frac{x^2}{x^2+2}dx$

Sol. $\int_{-1}^{2}\frac{x^2+2-1}{x^2+2}dx=\int_{-1}^{2}\left(1-\frac{2}{x^2+2}\right)dx$

$$=\int_{-1}^{2}dx-2\int_{-1}^{2}\frac{dx}{x^2+(\sqrt{2})^2}$$

$$=[x]_{-1}^{2}-2\cdot\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]^2$$

$$=[2-(-1)]-\sqrt{2}\left[\tan^{-1}\left(\frac{2}{\sqrt{2}}\right)-\tan^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right]$$

$$=3-\sqrt{2}\left[\tan^{-1}(\sqrt{2})-\tan^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right]$$

$$=3+\sqrt{2}\left[\tan^{-1}\left(-\frac{1}{\sqrt{2}}\right)-\tan^{-1}(\sqrt{2})\right]$$

3. $\int_{0}^{1}\frac{x^2}{x^2+1}dx$

Sol. $\int_{0}^{1}\frac{x^2}{x^2+1}dx=\int_{0}^{1}\frac{x^2+1-1}{x^2+1}dx=\int_{0}^{1}dx-\int_{0}^{1}\frac{dx}{x^2+1}$

$$[x]_0^1-\left[\tan^{-1}x\right]_0^1=1-\tan^{-1}1=1-\frac{\pi}{4}$$

4. $\int_{0}^{\pi/2}x^2\sin x dx$

Sol. $\int_{0}^{\pi/2}x^2\sin x dx=\left[x^2(-\cos x)\right]_0^{\pi/2}-\int_{0}^{\pi/2}(2x)(-\cos x)dx$

$$=(0-0)+2\int_{0}^{\pi/2}x\cos x\,dx$$

$$= 2\left[x\sin x\right]_0^{\pi/2} - \int_0^{\pi/2}(2)(\sin x)dx$$

$$= 2\left[\frac{\pi}{2}\times 1\right] + 2\left[\cos x\right]_0^{\pi/2}$$

$$= \pi + 2(0-1) = \pi - 2$$

5. $\int_0^4 |2-x|\,dx$

Sol. $\int_0^2 |2-x|\,dx + \int_2^4 |2-x|\,dx$

$$= \int_0^2 (2-x)dx + \int_2^4 (x-2)dx$$

$$= \left[2x - \frac{x^2}{2}\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$= \left(4 - \frac{4}{2}\right) - \left[(8-8) - \left(4 - \frac{4}{2}\right)\right]$$

$$= 2 - 0 + 2 = 4$$

6. $\int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x}dx$

Sol. Let $I = \int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x}dx$ **…(1)**

$$I = \int_0^{\pi/2} \frac{\sin^5(\pi/2 - x)dx}{\sin^5(\pi/2-x) + \cos^5(\pi/2-x)}\left(\because \int_0^a f(a-x)dx = \int_0^a f(x)\right)$$

$$= \int_0^{\pi/2} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x} \quad \text{…………(2)}$$

Adding (1) and (2) ,

$$2I = \int_0^{\pi/2} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x}dx = \int_0^{\pi/2} 1\cdot dx$$

$$2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

7. $\int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x} dx$

Sol. let $I = \int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x} dx$ **…(1)**

$$I = \int_0^{\pi/2} \frac{\sin^2(\pi/2 - x) - \cos^2(\pi/2 - x)}{\sin^3(\pi/2 - x) + \cos^3(\pi/2 - x)} dx \quad \left(\because \int_0^a f(a-x)dx = \int_0^a f(x)dx \right)$$

$$I = \int_0^{\pi/2} \frac{\cos^2 x - \sin^2 x}{\cos^3 x + \sin^3 x} dx \quad \textbf{…(2)}$$

Adding (1) and (2),

$$2I = \int_0^{\pi/2} \frac{0dx}{\cos^3 x + \sin^3 x} dx \Rightarrow I = 0$$

III. Evaluate the following integrals.

1. $\int_0^{\pi/2} \frac{dx}{4 + 5\cos x}$

Sol. $\int_0^{\pi/2} \frac{dx}{4 + 5\cos x} = \int_0^{\pi/2} \frac{dx}{4 + 5\left[\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right]}$

$$= \int_0^{\pi/2} \frac{dx}{\frac{4\left[\tan^2 \frac{x}{2} + 1\right] + 5\left[1 - \tan^2 \frac{x}{2}\right]}{\left[\tan^2 \frac{x}{2} + 1\right]}}$$

put $\tan \frac{x}{2} = t \quad \Rightarrow \frac{1}{2}\sec^2 \frac{x}{2} dx = dt \Rightarrow dx = \frac{2dt}{1 + t^2}$

$x = 0 \Rightarrow t = o$ and $x = \frac{\pi}{2} \Rightarrow t = 1$

$$= \int_0^1 \frac{\left(1+t^2\right)}{4t^2+4+5-5t^2}\frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{2}{9-t^2}dt = \frac{2}{2\cdot 3}\ln\left[\left|\frac{3+t}{3-t}\right|\right]_0^1$$

$$= \frac{1}{3}\left[\ln\frac{4}{2}\right] = \frac{1}{3}\ln 2$$

2. $\int_a^b \sqrt{(x-a)(b-x)}dx$

Sol. $\int_a^b \sqrt{(x-a)(b-x)}dx = \int_a^b \sqrt{-x^2+(a+b)x-ab}dx$

$$= \int_a^b \sqrt{\left(\frac{b-a}{2}\right)^2 - \left[x-\left(\frac{a+b}{2}\right)\right]^2}dx$$

$$\left(\because -x^2+(a+b)x-ab = -\left(x^2-(a+b)x+ab\right) = -\left(\left(x-\frac{a+b}{2}\right)^2 - \left(\frac{a+b}{2}\right)^2 + ab\right) = \left(\frac{b-a}{2}\right)^2 - \left[x-\left(\frac{a+b}{2}\right)\right]^2\right)$$

$$= \left[\frac{1}{2}\left(x-\left(\frac{a+b}{2}\right)\right)\sqrt{(x-a)(b-x)} + \frac{(b-a)^2}{4\cdot 2}\sin^{-1}\frac{\left(x-\left(\frac{a+b}{2}\right)\right)}{\left(\frac{b-a}{2}\right)}\right]_a^b$$

$$= 0 + \frac{(b-a)^2}{8}\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= \frac{(b-a)^2}{8}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = \frac{\pi}{8}(b-a)^2$$

3. $\int_0^{1/2} \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

put $\sin^{-1}x = t \Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dt$

Sol. and x=sint

$x=0 \Rightarrow t=0$ and $x=\frac{1}{2} \Rightarrow t=\frac{\pi}{6}$

$$\int_0^{1/2} \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx = \int_0^{\frac{\pi}{6}} t.\sin t dt = \left(t\int \sin t dt\right)_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} 1.(-\cos t)dt$$

$$= t(-\cos t)_o^{\frac{\pi}{6}} + (\sin t)_o^{\frac{\pi}{6}} = \frac{\pi}{6}\left(-\frac{\sqrt{3}}{2}\right) - 0 + \frac{1}{2} - 0$$

$$= \frac{1}{2} - \pi\frac{\sqrt{3}}{12}$$

4. $\int_0^{\pi/4} \frac{\sin x + \cos x}{9+16\sin 2x}dx$

Sol. $\int_0^{\pi/4} \frac{\sin x + \cos x}{9+16\sin 2x}dx = \int_0^{\pi/4} \frac{\sin x + \cos x}{9+16[1-(\sin x - \cos x)^2]}dx$

put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)dx = dt$

$x = 0 \Rightarrow t = -1$ and $x=\frac{\pi}{4} \Rightarrow t = 0$

$$= \int_{-1}^{0} \frac{dt}{25-16t^2} = \frac{1}{16}\int_{-1}^{0} \frac{dt}{\frac{25}{16} - t^2}$$

$$= \frac{1}{16} \times \frac{1}{2\times\frac{5}{4}}\left[\ln\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|\right]_{-1}^{0}$$

$$= -\frac{1}{40}\ln\left[\frac{1/4}{9/4}\right] = \frac{1}{40}\cdot 2\ln\cdot 3 = \frac{1}{20}\ln 3$$

5. $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x} dx$

Sol.

let $I = \int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x} dx$ **----(1)**

$= \int_0^{\pi/2} \frac{a\sin\left(\frac{\pi}{2} - x\right) + b\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} \left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$I = \int_0^{\pi/2} \frac{a\cos x + b\sin x}{\sin x + \cos x} dx$ **-------(2)**

(1)+(2) $\Rightarrow 2I = \int_0^{\pi/2} \frac{a(\sin x + \cos x) + b(\sin x + \cos x)}{\cos x + \sin x} dx$

$= \int_0^{\pi/2} (a+b)dx = (a+b)\frac{\pi}{2} \Rightarrow I = (a+b)\frac{\pi}{4}$

6. $\int_0^a x(a-x)^n dx$

Sol. let $I = \int_0^a x(a-x)^n dx$ **…(1)**

$I = \int_0^a (a-x)(x)^n dx$ **…(2)**

$I = \int_0^a ax^n dx - x^{n+1} dx$

$= \left[\frac{ax^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^a = \frac{a^{n+2}}{n+1} - \frac{a^{n+2}}{n+2}$

$I = \frac{a^{n+2}}{(n+1)(n+2)}$

7. $\int_0^2 x\sqrt{2-x}\,dx$

$\text{Ans}: \dfrac{16\sqrt{2}}{15}$

8. $\int_0^\pi x\sin^3 x dx$

Sol. $I = \int_0^\pi x\sin^3 x dx$

$$= \int_0^\pi (\pi - x)\sin^3(\pi - x)dx \qquad \left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$I = \int_0^\pi (\pi - x)\sin^3 x dx = \int_0^\pi \pi \sin^3 x dx - \int_0^\pi x\sin^3 x dx$$

$$= \int_0^\pi \pi\sin^3 x dx = I$$

$$\Rightarrow 2I = \int_0^\pi \pi\sin^3 x dx = \pi\int_0^\pi \frac{3\sin x - \sin 3x}{4}dx$$

$$= \frac{\pi}{4}\left(-3\cos x + \frac{\cos 3x}{3}\right)_0^\pi = \frac{\pi}{4}\left(-3.-1-\frac{1}{3}+3-\frac{1}{3}\right)$$

$$= \frac{\pi}{4}(6-2/3) = \frac{\pi}{4}.16/3$$

$$\therefore I = \frac{\pi}{2.4}\frac{16}{3} = \frac{2\pi}{3}$$

9. $\int_0^\pi \frac{x}{1+\sin x}dx$

Sol. $I = \int_0^\pi \frac{x}{1+\sin x}dx \quad$ **…(i)**

$$I = \int_0^\pi \frac{(\pi - x)dx}{1+\sin(\pi - x)} = \int_0^\pi \frac{\pi dx}{1+\sin x} - \int_0^\pi \frac{xdx}{1+\sin x}$$

$$= \int_0^\pi \frac{\pi dx}{1+\sin x} - I$$

$$2I=\int_0^{\pi}\frac{\pi dx}{1+\sin x}\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{1+\sin x}$$

$$=\frac{\pi}{2}\int_0^{\pi}\frac{(1-\sin x)}{1-\sin^2 x}dx=\frac{\pi}{2}\int_0^{\pi}\left(\frac{1-\sin x}{\cos^2 x}\right)dx$$

$$=\frac{\pi}{2}\left(\int_0^{\pi}\frac{1}{\cos^2 x}-\int_0^{\pi}\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}dx\right)$$

$$=\frac{\pi}{2}\int_0^{\pi}\sec^2 x dx-\int_0^{\pi}\sec x\cdot\tan x dx$$

$$=\frac{\pi}{2}\left([\tan x]_0^{\pi}-[\sec x]_0^{\pi}\right)$$

$$=\frac{\pi}{2}[(0-0)-(-1-1)]=\frac{\pi}{2}\cdot 2=\pi$$

10. $\displaystyle\int_0^{\pi}\frac{x\sin^3 x}{1+\cos^2 x}dx$

Sol. $\displaystyle I=\int_0^{\pi}\frac{x\sin^3 x}{1+\cos^2 x}dx$

$$=\int_0^{\pi}\frac{(\pi-x)\sin^3(\pi-x)}{1+\cos^2(\pi-x)}dx$$

$$=\int_0^{\pi}\frac{(\pi-x)\sin^3 x}{1+\cos^2 x}dx$$

$$=\pi\int_0^{\pi}\frac{\sin^3 x}{1+\cos^2 x}dx-\int_0^{\pi}\frac{x\sin^3 x}{1+\cos^2 x}dx$$

$$=\pi\int_0^{\pi}\frac{\sin^3 x}{1+\cos^2 x}dx-I$$

$$2I=\int_0^{\pi}\frac{\sin^3 x dx}{1+\cos^2 x}$$

Put t = cos x ⇒ dt = –sin x dx

$$2I=\int_1^{-1}-\frac{(1-t^2)}{1+t^2}dt=\int_{-1}^{1}\frac{1-t^2}{1+t^2}dt$$

$$= \int_{-1}^{1}\left(-1+\frac{2}{1+t^2}\right)dt = \left[-t+2\tan^{-1}t\right]_{-1}^{1}$$

$$= \left[-1+2\tan^{-1}1\right]-\left[-1+2\tan^{-1}(-1)\right]$$

$$= -1+2\cdot\frac{\pi}{4}+1-2\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

$$I=\frac{\pi}{2}$$

11. $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

Sol. $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

Put x = tan θ

$dx = \sec^2\theta\, d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \pi/4$

$$I = \int_0^1 \frac{\log(1+x)}{1+x^2}dx = \int_0^{\pi/4}\frac{\log(1+\tan\theta)\sec^2\theta d\theta}{(1+\tan^2\theta)}$$

$$= \int_0^{\pi/4}\log(1+\tan\theta)d\theta$$

$$\text{let } I = \int_0^{\pi/4}\log(1+\tan\theta)d\theta$$

$$= \int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$$

$$= \int_0^{\frac{\pi}{4}}\log\left[1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right]d\theta$$

$$= \int_0^{\pi/4}\log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$= \int_0^{\pi/4}\log\left[\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$= \int_0^{\pi/4} \log \frac{2}{1+\tan\theta} d\theta$$

$$= \int_0^{\pi/4} [\log 2 - \log(1+\tan\theta)] d\theta$$

$$= \log 2 \int_0^{\pi/4} d\theta - \int_0^{\pi/4} \log(1+\tan\theta) d\theta$$

$$= \log 2 \int_0^{\pi/4} d\theta - I$$

$$2I = \log 2 (\theta)_0^{\pi/4} = (\log 2)\frac{\pi}{4}$$

$$\therefore I = \frac{\pi}{8} \log 2$$

12. $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

Sol. $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x) dx}{1+\cos^2(\pi - x)}$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx = \int_0^{\pi} \frac{\sin x dx}{1+\cos^2 x} - \int_0^{\pi} \frac{x \sin x dx}{1+\cos^2 x}$$

$$= \pi\left\{\tan^{-1}(-\cos x)\right\}_0^{\pi} - I$$

$$2I = \pi\left\{\tan^{-1} 1 - \tan^{-1}(-1)\right\} = \pi\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = 2\frac{\pi^2}{4}$$

$$I = \frac{\pi^2}{4} \Rightarrow \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}$$

13. $\int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx$

Sol. $I = \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx$ **----1.**

$$= \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\cos^2 x dx}{\sin x + \cos x} \text{ ----2.}$$

Adding 1. and 2.

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx$$

$$\Rightarrow I == \frac{1}{2} \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

Consider $\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

Put tan(x/2) = t

$$dx = \frac{2dt}{1+t^2}, \ \cos x = \frac{1-t^2}{1+t^2}, \sin x = \frac{2t}{1+t^2}$$

$$\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \int_0^1 \frac{2tdt}{2t + (1-t^2)}$$

$$= 2\int_0^1 \frac{dt}{(\sqrt{2})^2 - (t-1)^2} = 2 \cdot \frac{1}{2\sqrt{2}} \left[\log \frac{\sqrt{2} + t - 1}{\sqrt{2} - t + 1} \right]_0^1$$

$$= \frac{1}{\sqrt{2}} \left(\log 1 - \log \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right)$$

$$= \frac{1}{\sqrt{2}} \log \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}$$

$$= \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)^2 = \frac{2}{\sqrt{2}} \log(\sqrt{2} + 1)$$

$$I = \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$$

14. Suppose that f : R $\rightarrow$ R is a continuous periodic function and T is the period of it. Let a $\in$ R. Then prove that for any positive integer n, $\int_0^{a+nT} f(x)dx = n \int_0^{a+T} f(x)dx$**.**

Sol. $\int_0^{a+nT} f(x)dx = \int_0^{a+T} f(x)dx + \int_{a+T}^{a+2T} f(x)dx \ \ldots$ **sx** $+ \int_{a+rT}^{a+(r+1)T} f(x)dx + \ldots + \int_{a+(x-1)T}^{a+xT} f(x)dx$ **…(1)**

Consider (r+1)th integral of RHS

$$\int_{a+rT}^{a+(r+1)T} f(x)dx$$

Let x = y + rT $\Rightarrow$ dx = dy

x = a + rT ⇒ y = a

x = a + (r + 1)T ⇒ y = a + T

$$\int_{a+rT}^{a+(r+1)T} f(x)dx = \int_{a}^{a+T} f(y+rT)dy$$

$$= \int_{a}^{a+T} f(y)dy \quad \textbf{(f is periodic)}$$

$$= \int_{a}^{a+T} f(x)dx$$

Similarly we can show that each integral of (1) is equal to $\int_{a}^{a+T} f(x)dx$**.**

$$\therefore \int_{0}^{a+nT} f(x)dx = \int_{0}^{a+T} f(x)dx + \int_{0}^{a+T} f(x)dx \text{ ...n terms} = n\int_{0}^{a+T} f(x)dx$$

REDUCTION FORMULAE

THEOREM 1 :

If $I_n = \int_{0}^{\pi/2} \sin^n x\,dx$ **then** $I_n = \frac{n-1}{n} I_{n-2}$**.**

Proof :

$$I_n = \int_{0}^{\pi/2} \sin^n x\,dx = \int_{0}^{\pi/2} \sin^{n-1} x.\sin x\,dx$$

$$= \left[-\sin^{n-1} x\cos x\right]_0^{\pi/2} + \int_{0}^{\pi/2} (n-1)\sin^{n-2} x.\cos^2 x\,dx$$

$$= (n-1)\int_{0}^{\pi/2} \sin^{n-2} x(1-\sin^2 x)dx$$

$$= (n-1)\left[\int_{0}^{\pi/2} \sin^{n-2} x\,dx - \int_{0}^{\pi/2} \sin^n x\,dx\right]$$

$$= (n-1)I_{n-2} - (n-1)I_n$$

$$I_n(1+n-1) = (n-1)I_{n-2} \Rightarrow I_n n = (n-1)I_{n-2}$$

$$\therefore I_n = \frac{n-1}{n} I_{n-2} ------(1)$$

Note

In (1), replace n by n-2,n-3,------------- then

$$I_n = \frac{n-1}{n} I_{n-2} \Rightarrow I_{n-2} = \frac{n-3}{n-2} I_{n-4} \Rightarrow I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot ......I_0$ **or I_1 according as n is even or odd.**

But $I_0 = \int_0^{\pi/2} \sin^0 x\,dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$

$$I_1 = \int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2}$$

$$= -\cos\frac{\pi}{2} + \cos 0 = -0 + 1 = 1$$

$\therefore \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot .... \frac{1}{2} \cdot \frac{\pi}{2}$ **if n is even.**

$\therefore \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot .... \frac{2}{3} \cdot 1$ **if n is odd.**

THEOREM 2 :

If $I_n = \int_0^{\pi/2} \cos^n x\,dx$ **then** $I_n = \frac{n-1}{n} I_{n-2}$**.**

$$I_n = \int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \cos^n \left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \sin^n x\,dx$$

THEOREM 3 : If $I_n = \int_0^{\pi/4} \tan^n x\,dx$ **then** $I_n + I_{n-2} = \frac{1}{n-1}$**.**

Proof :

$$I_n = \int_0^{\pi/4} \tan^n x\,dx = \int_0^{\pi/4} \tan^{n-2} x \tan^2 x\,dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x(\sec^2 x - 1)dx = \int_0^{\pi/4} \tan^{n-2} x \sec^2 x\,dx - \int_0^{\pi/4} \tan^{n-2} x\,dx$$

$$= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - I_{n-2} = \frac{1}{n-1} - I_{n-2}$$

$$\therefore I_n + I_{n-2} = \frac{1}{n-1}$$

THEOREM 4 :

If $I_n = \int_0^{\pi/4} \sec^n x\,dx$ **then** $I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$ **.**

Proof :

$$I_n = \int_0^{\pi/4} \sec^n x\,dx = \int_0^{\pi/4} \sec^{n-2} x \sec^2 x\,dx$$

$$= \left[\sec^{n-2} x \tan x\right]_0^{\pi/4} - \int_0^{\pi/4} (n-2)\sec^{n-2} x \sec x \tan^2 x\,dx$$

$$= (\sqrt{2})^{n-2} - (n-2)\int_0^{\pi/4} \sec^{n-2} x(\sec^2 x - 1)\,dx$$

$$= (\sqrt{2})^{n-2} - (n-2)\left[\int_0^{\pi/4} \sec^n x\,dx - \int_0^{\pi/4} \sec^{n-2} x\,dx\right]$$

$$= (\sqrt{2})^{n-2} - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n(1+n-2) = (\sqrt{2})^{n-2} + (n-2)I_{n-2}$$

$$\Rightarrow I_n(n-1) = (\sqrt{2})^{n-2} + (n-2)I_{n-2}$$

$$\therefore I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

THEOREM 5: If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx$ **then** $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx$ **.**

Proof :

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx = \int_0^{\pi/2} \sin^{m-1}(\sin x \cos^n x)dx$$

$$= \left[\frac{-\sin^{m-1} x \cos^{n+1} x}{n+1}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos^{n+1} x}{n+1}(m-1)\sin^{m-2} x \cos x\,dx$$

$$= \frac{m-1}{n+1}\int_0^{\pi/2} \sin^{m-2} x \cos^n x \cos^2 x\,dx$$

$$= \frac{m-1}{n+1}\int_0^{\pi/2} \sin^{m-2} x \cos^n x(1-\sin^2 x)dx$$

$$= \frac{m-1}{n+1} \int_0^{\pi/2} \sin^{m-2} x \cos^n x - \sin^m x \cos^n x) dx$$

$$= \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n}$$

$$\Rightarrow I_{m,n} \left(1 + \frac{m-1}{n+1}\right) = \frac{m-1}{n+1} I_{m-2,n}$$

$$\Rightarrow I_{m,n} \left(\frac{n+m}{n+1}\right) = \frac{m-1}{n+1} I_{m-2,n}$$

$$\therefore I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$$ **-----(1)**

Note: replacing m by m-2,m-4,

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n} = \frac{m-1}{m+n} \frac{m-3}{m+n-2} I_{m-4,n} = \frac{m-1}{m+n} \frac{m-3}{m+n-2} \frac{m-5}{m+n-4} \ldots I_{0,n}$$

or $I_{1,n}$ according as n is even or odd.

But $I_{0,1} = \int_0^{\pi/2} \sin^0 x \cos^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx$

$$I_{1,n} = \int_0^{\pi/2} \sin x \cos^n x \, dx = \left[-\frac{\cos^{n+1} x}{n+1} \right]_0^{\pi/2} = \frac{1}{n+1}$$

$$\therefore I_{m,n} = \frac{m-1}{m+n} \frac{m-3}{m+n-2} \ldots \frac{1}{n+1} \text{ if m is odd}$$

$$= \frac{m-1}{m+n} \frac{m-3}{m+n-2} \ldots \int_0^{\pi/2} \cos^n x \, dx \text{ if m is even}$$

COROLLARY 2: If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$ **then** $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$.