

**CHAPTER 9**  
**DEFINITE INTEGRATION**

**TOPICS:**

- 1. The definite integral**
- 2. Interpretation of definite integral**
- 3. Fundamental theorem of integral calculus**
- 4. Properties of definite integral**
- 5. Reduction formulae**

## DEFINITE INTEGRAL

Let  $f(x)$  be a function defined on  $[a, b]$ . If  $\int f(x) dx = F(x)$ , then  $F(b) - F(a)$  is called the definite integral of  $f(x)$  over  $[a, b]$ . It is denoted by  $\int_a^b f(x) dx$ . The real number 'a' is called the lower limit and the real number 'b' is called the upper limit. This is known as fundamental theorem of integral calculus.

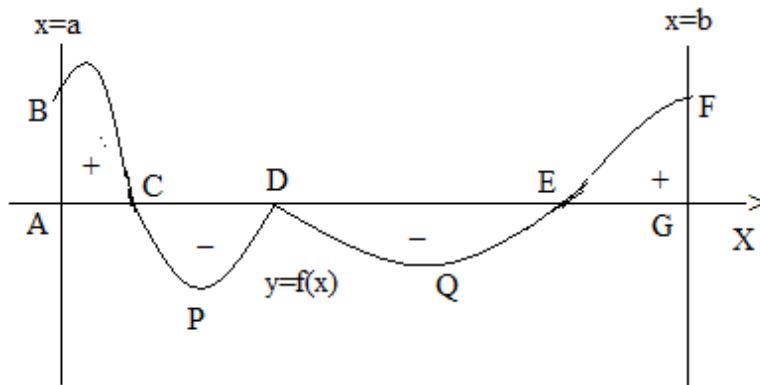
### THEOREM

$$\int_a^b f(x) dx = \int_a^b f(t) dt \quad \text{i.e., definite integral is independent of its variable.}$$

### Geometrical interpretation of definite integral

If  $f(x) > 0$  for all  $x$  in  $[a, b]$  then  $\int_a^b f(x) dx$  is numerically equal to the area bounded by the curve  $y = f(x)$ , the  $x$ -axis and the lines  $x=a$  and  $x=b$  i.e.,  $\int_a^b f(x) dx$ .

In general,  $\int_a^b f(x) dx$  represents to algebraic sum of the areas of the figures bounded by the curve  $y = f(x)$ , the  $x$  axis and the lines  $x=a$  and  $x=b$ . the areas above  $x$ -axis are taken with plus sign and the areas below  $x$ -axis are taken with minus sign i.e.,



$$\int_a^b f(x) dx = \text{area ABC} - \text{area CPD} - \text{area EQE} + \text{area EFG.}$$

## PROPERTIES OF DEFINITE INTEGRALS

$$1. \int_a^b f(x)dx = -\int_b^a f(x)dx.$$

**Proof :** let  $\int f(x)dx = F(x)$  then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) = -[F(a) - F(b)] = -[F(x)]_b^a = -\int_b^a f(x)dx$$

$$2. \text{ If } a < c < b \text{ then } \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

**Proof :** Let  $\int f(x)dx = F(x)$ .

$$\text{Then } \int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

$$\text{R.H.S.} = \int_a^c f(x)dx + \int_c^b f(x)dx = [F(x)]_a^c + [F(x)]_c^b$$

$$= F(c) - F(a) + F(b) - F(c) = F(b) - F(a) = \int_a^b f(x)dx = \text{L.H.S}$$

$$3. \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

**Proof :** Put  $a - x = t. \Rightarrow dx = -dt$

**L.L, x = 0**  $\Rightarrow t = a$

**U.L. x=a**  $\Rightarrow t=0$ .

$$\text{R.H.S.} = \int_0^a f(a-x)dx = \int_a^0 f(t)(-dt) = -\int_a^0 f(t)dt = \int_0^a f(t)dt = \int_0^a f(x)dx = \text{L.H.S.}$$

### THEOREM

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

**Proof :** Put  $a + b - x = t$ , then  $-dx = dt \Rightarrow dx = -dt$

**L.L, x = a**  $\Rightarrow t = b$

**U.L. x=b**  $\Rightarrow t=a$ .

$$\text{R.H.S.} = \int_a^b f(a+b-x)dx = \int_b^a f(t)(-dt) = -\int_b^a f(t)dt = \int_a^b f(t)dt = \int_a^b f(x)dx = \text{LHS}$$

**THEOREM**

$$\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx, \text{ if } f(x) \text{ is an even function} = 0, \text{ if } f(x) \text{ is an odd function.}$$

**Proof :** Since  $-a < 0 < a$ ,  $\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$

In the 1<sup>st</sup> integral of RHS,

Put  $x = -t$ , then  $dx = -dt$ ,

L.L,  $x = -a \Rightarrow t = a$

U.L.  $x=0 \Rightarrow t=0$ .

$$\begin{aligned} \therefore \int_{-a}^a f(x)dx &= \int_a^0 f(-t)(-dt) + \int_0^a f(x)dx = -\int_a^0 f(-t)dt + \int_0^a f(x)dx \\ &= \int_0^a f(-t)dt + \int_0^a f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx \text{ -----(1)} \end{aligned}$$

**Case I : If  $f(x)$  is an even function then  $f(-x) = f(x)$**

Then from (1),

$$\int_{-a}^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2 \int_0^a f(x)dx$$

**Case II :If  $f(x)$  is an odd function then  $f(-x) = -f(x)$**

From (1),

$$\therefore \int_{-a}^a f(x)dx = -\int_0^a f(x)dx + \int_0^a f(x)dx = 0$$

**THEOREM**

$$\int_0^{2a} f(x)dx = 2 \int_0^a f(x)dx \text{ if } f(2a - x) = f(x) \text{ or } 0 \text{ if } f(2a - x) = -f(x).$$

**Proof :**

Since  $0 < a < 2a$ ,  $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx \text{ ---(1)}$

In the 2<sup>nd</sup> integral of rhs, Put  $2a - t = x$ , then  $-dx = dt \Rightarrow dx = -dt$

**L.L,  $x = a \Rightarrow t = a$**

**U.L.  $x=2a \Rightarrow t=0$ .**

$$\int_a^{2a} f(x)dx = \int_a^0 f(2a-t)(-dt) = -\int_a^0 f(2a-t)dt = \int_0^a f(2a-t)dt = \int_0^a f(2a-x)dx$$

**From (1),**

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx \text{ ----(2)}$$

**Case I : if  $f(2a - x) = f(x)$**

**From (2),**

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2 \int_0^a f(x)dx$$

**Case II : if  $f(2a - x) = -f(x)$**

**From (2),**

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx = 0$$

**THEOREM :**

**If  $f(x)$  is a periodic function with period 'T' then  $\int_0^{nT} f(x)dx = n \int_0^T f(x)dx$ .**

**Proof : Let  $S(n)$  be the statement that  $\int_0^{nT} f(x)dx = n \int_0^T f(x)dx$  for  $n \in \mathbb{N}$ .**

**Let  $n=1$ , then**

$$\int_0^{1T} f(x)dx = \int_0^T f(x)dx = 1 \int_0^T f(x)dx$$

**$\therefore S(1)$  is true.**

**Assume that  $S(k)$  is true.**

$$\therefore \int_0^{kT} f(x)dx = k \int_0^T f(x)dx$$

$$\text{Now } \int_0^{(k+1)T} f(x)dx = \int_0^{kT} f(x)dx + \int_{kT}^{(k+1)T} f(x)dx = k \int_0^T f(x)dx + \int_{kT}^{(k+1)T} f(x)dx$$

**In the 2<sup>nd</sup> integral of rhs**

Put  $x = kT + t$ , then  $dx = dt$ .

$$x = kT, (k + 1)T \Rightarrow t = 0, t = a.$$

$$\int_{kT}^{(k+1)T} f(x)dx = \int_0^T f(ka + t)dt = \int_0^T f(t)dt = \int_0^T f(x)dx$$

[ $\because f(x)$  is a periodic function with period 'T']

$$\begin{aligned} \therefore \int_0^{(k+1)T} f(x)dx &= k \int_0^T f(x)dx + \int_0^T f(x)dx \\ &= k \int_0^T f(x)dx + \int_0^T f(x)dx = (k+1) \int_0^T f(x)dx \end{aligned}$$

$\therefore S(k + 1)$  is true. By principle of Mathematical Induction  $S(n)$  is true,  $\forall n \in \mathbb{N}$ .

$$\therefore \int_0^{nT} f(x)dx = n \int_0^T f(x)dx .$$

### THEOREM

If  $f(x)$  is an integrable function on  $[a, b]$  and  $g(x)$  is derivable on  $[a, b]$  then

$$\int_a^b (f \circ g)(x)g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx .$$

### EXERCISE – 9(A)

I. Evaluate the following definite integrals.

1.  $\int_0^a (a^2x - x^3)dx$

Sol.  $\int_0^a (a^2x - x^3)dx = \left[ \frac{a^2x^2}{2} - \frac{x^4}{4} \right]_0^a = \frac{a^4}{2} - \frac{a^4}{4} = \frac{a^4}{4}$

2.  $\int_2^3 \frac{2xdx}{1+x^2}$

Sol.  $\int_2^3 \frac{2xdx}{1+x^2} = \left[ \ln |1+x^2| \right]_2^3 = \ln 10 - \ln 5 = \ln(10/5) = \ln 2$

$$3. \int_0^{\pi} \sqrt{2+2\cos\theta} d\theta$$

$$\begin{aligned} \text{Sol. } \int_0^{\pi} \sqrt{2+2\cos\theta} d\theta &= \int_0^{\pi} \sqrt{2} \cdot \sqrt{2} \sqrt{\cos^2 \frac{\theta}{2}} d\theta = \int_0^{\pi} 2 \cos(\theta/2) d\theta \\ &= \left[ 4 \sin \frac{\theta}{2} \right]_0^{\pi} = 4 \left( \sin \frac{\pi}{2} - \sin 0 \right) = 4 \end{aligned}$$

$$4. \int_0^{\pi} \sin^3 x \cdot \cos^3 x dx$$

$$\begin{aligned} \text{Sol. } \int_0^{\pi} \sin^3 x \cdot \cos^3 x dx &= \int_0^{\pi} \sin^3(\pi-x) \cos^3(\pi-x) dx \\ &= - \int_0^{\pi} \sin^3 x \cos^3 x dx = -I \\ \Rightarrow 2I &= 0 \Rightarrow I = 0 = -I \\ 2I &= 0 \Rightarrow I = 0 \end{aligned}$$

$$5. \int_0^2 |1-x| dx$$

$$\begin{aligned} \text{Sol. } \int_0^2 |1-x| dx &= \int_0^1 -(x-1) dx + \int_1^2 (x-1) dx \\ &= \int_0^1 (-x+1) dx + \int_1^2 (x-1) dx \\ &= \left[ \frac{-x^2}{2} + x \right]_0^1 + \left[ \frac{x^2}{2} - x \right]_1^2 \\ &= -\frac{1}{2} + 1 + \left( \frac{4}{2} - 2 \right) - \left( \frac{1}{2} - 1 \right) = \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

$$6. \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$$

$$\text{Sol. Let } I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx \quad \dots(i)$$

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos(\pi/2 - \pi/2 - x) dx}{1 + e^{-x}} \left( \because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right)$$

$$= \int_{-\pi/2}^{\pi/2} \frac{e^x \cos x dx}{1 + e^x} \text{-----(2)}$$

**Adding (1) and (2) ,**

$$2I = \int_{-\pi/2}^{\pi/2} \frac{\cos x(1 + e^x)}{1 + e^x} dx = \int_{-\pi/2}^{\pi/2} \cos x dx$$

$$2I = 2 \int_0^{\pi/2} \cos x dx \quad (\because \cos x \text{ is even function})$$

$$\Rightarrow I = [\sin x]_0^{\pi/2} \Rightarrow I = 1$$

7.  $\int_0^1 \frac{dx}{\sqrt{3-2x}}$

**Sol.**

$$\int_0^1 \frac{dx}{\sqrt{3-2x}} = \left( \frac{2\sqrt{3-2x}}{-2} \right)_0^1 = -(\sqrt{3-2 \cdot 1} - \sqrt{3-2 \cdot 0}) = -(1 - \sqrt{3}) = (\sqrt{3} - 1)$$

8.  $\int_0^a (\sqrt{a} - \sqrt{x})^2 dx$

**Sol.**  $\int_0^a (\sqrt{a} - \sqrt{x})^2 dx = \int_0^a (a + x + 2\sqrt{a}\sqrt{x}) dx$

$$= \left[ ax + \frac{x^2}{2} - 2\sqrt{a} \cdot x^{3/2} \cdot \frac{2}{3} \right]_0^a$$

$$a^2 + \frac{a^2}{2} - \frac{4}{3}a^2 = \frac{6a^2 + 3a^2 - 8a^2}{6} = \frac{1}{6}a^2$$



9.  $\int_0^{\pi/4} \sec^4 \theta d\theta$

$$\int_0^{\pi/4} \sec^4 \theta d\theta = \int_0^{\pi/4} \sec^2 \theta \cdot \sec^2 \theta d\theta = \int_0^{\pi/4} \sec^2 \theta (1 + \tan^2 \theta) d\theta$$

**Sol. Let**  $= \int_0^{\pi/4} (\sec^2 \theta + \sec^2 \theta \tan^2 \theta) d\theta = \int_0^{\pi/4} \sec^2 \theta d\theta + \int_0^{\pi/4} \tan^2 \theta \sec^2 \theta d\theta$

$$= \tan \theta \Big|_0^{\pi/4} + \left( \frac{\tan^3 \theta}{3} \right) \Big|_0^{\pi/4} = 1 - 0 + \frac{1}{3}(1 - 0) = \frac{4}{3}$$

10.  $I = \int_0^3 \frac{x}{\sqrt{x^2 + 16}} dx$

**Asn: 1**

11.  $\int_0^1 x \cdot e^{-x^2} dx$

**Sol.**  $\int_0^1 x \cdot e^{-x^2} dx = \frac{1}{2} \int_0^1 2x e^{-x^2} dx$ , put  $-x^2 = t$

$$\Rightarrow -2x dx = dt \Rightarrow 2x dx = -dt$$

$$x = 1 \Rightarrow t = -1, x = 0 \Rightarrow t = 0$$

$$I = \frac{1}{2} \int_0^{-1} -e^t dt = \frac{1}{2} [-e^t]_0^{-1}$$

$$= \frac{1}{2} [e^0 - e^{-1}] = \frac{1}{2} \left( 1 - \frac{1}{e} \right)$$

12.  $I = \int_1^5 \frac{dx}{\sqrt{2x-1}}$

**Ans:2**

**II. Evaluate the following integrals.**

1.  $\int_0^4 \frac{x^2}{1+x} dx$

**Sol.**  $\int_0^4 \frac{x^2}{1+x} dx = \int_0^4 \frac{x^2 - 1 + 1}{1+x} dx \Rightarrow I = \int_0^4 (x-1) dx + \int_0^4 \frac{dx}{1+x}$

$$\begin{aligned}
&= \left[ \frac{x^2}{2} - x \right]_0^4 + [\log(1+x)]_0^4 \\
&= \frac{4^2}{2} - 4 + \log 5 - \log 1 = 4 + \log 5
\end{aligned}$$

2.  $\int_{-1}^2 \frac{x^2}{x^2+2} dx$

**Sol.**  $\int_{-1}^2 \frac{x^2+2-1}{x^2+2} dx = \int_{-1}^2 \left( 1 - \frac{2}{x^2+2} \right) dx$

$$\begin{aligned}
&= \int_{-1}^2 dx - 2 \int_{-1}^2 \frac{dx}{x^2+(\sqrt{2})^2} \\
&= [x]_{-1}^2 - 2 \cdot \frac{1}{\sqrt{2}} \left[ \tan^{-1} \left( \frac{x}{\sqrt{2}} \right) \right]_{-1}^2 \\
&= [2 - (-1)] - \sqrt{2} \left[ \tan^{-1} \left( \frac{2}{\sqrt{2}} \right) - \tan^{-1} \left( -\frac{1}{\sqrt{2}} \right) \right] \\
&= 3 - \sqrt{2} \left[ \tan^{-1}(\sqrt{2}) - \tan^{-1} \left( -\frac{1}{\sqrt{2}} \right) \right] \\
&= 3 + \sqrt{2} \left[ \tan^{-1} \left( -\frac{1}{\sqrt{2}} \right) - \tan^{-1}(\sqrt{2}) \right]
\end{aligned}$$

3.  $\int_0^1 \frac{x^2}{x^2+1} dx$

**Sol.**  $\int_0^1 \frac{x^2}{x^2+1} dx = \int_0^1 \frac{x^2+1-1}{x^2+1} dx = \int_0^1 dx - \int_0^1 \frac{dx}{x^2+1}$

$$[x]_0^1 - [\tan^{-1} x]_0^1 = 1 - \tan^{-1} 1 = 1 - \frac{\pi}{4}$$

4.  $\int_0^{\pi/2} x^2 \sin x dx$

**Sol.**  $\int_0^{\pi/2} x^2 \sin x dx = [x^2(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} (2x)(-\cos x) dx$

$$= (0-0) + 2 \int_0^{\pi/2} x \cos x dx$$

$$\begin{aligned}
&= 2 \left[ x \sin x \right]_0^{\pi/2} - \int_0^{\pi/2} (2)(\sin x) dx \\
&= 2 \left[ \frac{\pi}{2} \times 1 \right] + 2 \left[ \cos x \right]_0^{\pi/2} \\
&= \pi + 2(0 - 1) = \pi - 2
\end{aligned}$$

5.  $\int_0^4 |2 - x| dx$

Sol.  $\int_0^2 |2 - x| dx + \int_2^4 |2 - x| dx$

$$\begin{aligned}
&= \int_0^2 (2 - x) dx + \int_2^4 (x - 2) dx \\
&= \left[ 2x - \frac{x^2}{2} \right]_0^2 + \left[ \frac{x^2}{2} - 2x \right]_2^4 \\
&= \left( 4 - \frac{4}{2} \right) - \left[ (8 - 8) - \left( 4 - \frac{4}{2} \right) \right] \\
&= 2 - 0 + 2 = 4
\end{aligned}$$

6.  $\int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx$

Sol. Let  $I = \int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx \quad \dots(1)$

$$\begin{aligned}
I &= \int_0^{\pi/2} \frac{\sin^5(\pi/2 - x) dx}{\sin^5(\pi/2 - x) + \cos^5(\pi/2 - x)} \left( \because \int_0^a f(a-x) dx = \int_0^a f(x) dx \right) \\
&= \int_0^{\pi/2} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x} \quad \dots\dots\dots(2)
\end{aligned}$$

**Adding (1) and (2) ,**

$$2I = \int_0^{\pi/2} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx = \int_0^{\pi/2} 1 \cdot dx$$

$$2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

$$7. \int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x} dx$$

$$\text{Sol. let } I = \int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x} dx \dots(1)$$

$$I = \int_0^{\pi/2} \frac{\sin^2(\pi/2 - x) - \cos^2(\pi/2 - x)}{\sin^3(\pi/2 - x) + \cos^3(\pi/2 - x)} dx \quad \left( \because \int_0^a f(a-x) dx = \int_0^a f(x) dx \right)$$

$$I = \int_0^{\pi/2} \frac{\cos^2 x - \sin^2 x}{\cos^3 x + \sin^3 x} dx \dots(2)$$

**Adding (1) and (2),**

$$2I = \int_0^{\pi/2} \frac{0 dx}{\cos^3 x + \sin^3 x} dx \Rightarrow I = 0$$

**III. Evaluate the following integrals.**

$$1. \int_0^{\pi/2} \frac{dx}{4 + 5 \cos x}$$

$$\text{Sol. } \int_0^{\pi/2} \frac{dx}{4 + 5 \cos x} = \int_0^{\pi/2} \frac{dx}{4 + 5 \left[ \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right]}$$

$$= \int_0^{\pi/2} \frac{dx}{4 \left[ \frac{\tan^2 \frac{x}{2} + 1}{\tan^2 \frac{x}{2} + 1} \right] + 5 \left[ \frac{1 - \tan^2 \frac{x}{2}}{\tan^2 \frac{x}{2} + 1} \right]}$$

$$\text{put } \tan \frac{x}{2} = t \Rightarrow \frac{1}{2} \sec^2 \frac{x}{2} dx = dt \Rightarrow dx = \frac{2dt}{1+t^2}$$

$$x = 0 \Rightarrow t = 0 \quad \text{and } x = \frac{\pi}{2} \Rightarrow t = 1$$

$$\begin{aligned}
&= \int_0^1 \frac{(1+t^2)}{4t^2+4+5-5t^2} \frac{2dt}{1+t^2} \\
&= \int_0^1 \frac{2}{9-t^2} dt = \frac{2}{2 \cdot 3} \ln \left[ \left| \frac{3+t}{3-t} \right| \right]_0^1 \\
&= \frac{1}{3} \left[ \ln \frac{4}{2} \right] = \frac{1}{3} \ln 2
\end{aligned}$$

2.  $\int_a^b \sqrt{(x-a)(b-x)} dx$

Sol.  $\int_a^b \sqrt{(x-a)(b-x)} dx = \int_a^b \sqrt{-x^2 + (a+b)x - ab} dx$

$$= \int_a^b \sqrt{\left(\frac{b-a}{2}\right)^2 - \left[x - \left(\frac{a+b}{2}\right)\right]^2} dx$$

$$\left( \begin{aligned}
&\because -x^2 + (a+b)x - ab = -(x^2 - (a+b)x + ab) = -\left(\left(x - \frac{a+b}{2}\right)^2 - \left(\frac{a+b}{2}\right)^2 + ab\right) \\
&= \left(\frac{b-a}{2}\right)^2 - \left[x - \left(\frac{a+b}{2}\right)\right]^2
\end{aligned} \right)$$

$$= \left[ \frac{1}{2} \left(x - \left(\frac{a+b}{2}\right)\right) \sqrt{(x-a)(b-x)} + \frac{(b-a)^2}{4 \cdot 2} \sin^{-1} \frac{\left(x - \left(\frac{a+b}{2}\right)\right)}{\left(\frac{b-a}{2}\right)} \right]_a^b$$

$$= 0 + \frac{(b-a)^2}{8} \left[ \sin^{-1}(1) - \sin^{-1}(-1) \right]$$

$$= \frac{(b-a)^2}{8} \left[ \frac{\pi}{2} + \frac{\pi}{2} \right] = \frac{\pi}{8} (b-a)^2$$

$$3. \int_0^{1/2} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

$$\text{put } \sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$$

**Sol.** and  $x = \sin t$

$$x=0 \Rightarrow t=0 \text{ and } x=\frac{1}{2} \Rightarrow t = \frac{\pi}{6}$$

$$\int_0^{1/2} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\pi/6} t \cdot \sin t dt = \left( t \int \sin t dt \right)_0^{\pi/6} - \int_0^{\pi/6} 1 \cdot (-\cos t) dt$$

$$= t(-\cos t) \Big|_0^{\pi/6} + (\sin t) \Big|_0^{\pi/6} = \frac{\pi}{6} \left( -\frac{\sqrt{3}}{2} \right) - 0 + \frac{1}{2} - 0$$

$$= \frac{1}{2} - \frac{\pi\sqrt{3}}{12}$$

$$4. \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

$$\text{Sol. } \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16[1 - (\sin x - \cos x)^2]} dx$$

$$\text{put } \sin x - \cos x = t \Rightarrow (\cos x + \sin x) dx = dt$$

$$x = 0 \Rightarrow t = -1 \text{ and } x = \frac{\pi}{4} \Rightarrow t = 0$$

$$= \int_{-1}^0 \frac{dt}{25 - 16t^2} = \frac{1}{16} \int_{-1}^0 \frac{dt}{\frac{25}{16} - t^2}$$

$$= \frac{1}{16} \times \frac{1}{2 \times \frac{5}{4}} \left[ \ln \left| \frac{\frac{5}{4} + t}{\frac{5}{4} - t} \right| \right]_{-1}^0$$

$$= -\frac{1}{40} \ln \left[ \frac{1/4}{9/4} \right] = \frac{1}{40} \cdot 2 \ln \cdot 3 = \frac{1}{20} \ln 3$$

5.  $\int_0^{\pi/2} \frac{a \sin x + b \cos x}{\sin x + \cos x} dx$

**Sol.**

$$\text{let } I = \int_0^{\pi/2} \frac{a \sin x + b \cos x}{\sin x + \cos x} dx \text{ ----(1)}$$

$$= \int_0^{\pi/2} \frac{a \sin\left(\frac{\pi}{2} - x\right) + b \cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx \left( \because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$I = \int_0^{\pi/2} \frac{a \cos x + b \sin x}{\sin x + \cos x} dx \text{ -----(2)}$$

$$(1)+(2) \Rightarrow 2I = \int_0^{\pi/2} \frac{a(\sin x + \cos x) + b(\sin x + \cos x)}{\cos x + \sin x} dx$$

$$= \int_0^{\pi/2} (a+b) dx = (a+b) \frac{\pi}{2} \Rightarrow I = (a+b) \frac{\pi}{4}$$

6.  $\int_0^a x(a-x)^n dx$

**Sol.** let  $I = \int_0^a x(a-x)^n dx \quad \dots(1)$

$$I = \int_0^a (a-x)(x)^n dx \quad \dots(2)$$

$$I = \int_0^a ax^n dx - x^{n+1} dx$$

$$= \left[ \frac{ax^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right]_0^a = \frac{a^{n+2}}{n+1} - \frac{a^{n+2}}{n+2}$$

$$I = \frac{a^{n+2}}{(n+1)(n+2)}$$

$$7. \int_0^2 x\sqrt{2-x} dx$$

$$\text{Ans: } \frac{16\sqrt{2}}{15}$$

$$8. \int_0^{\pi} x \sin^3 x dx$$

$$\text{Sol. } I = \int_0^{\pi} x \sin^3 x dx$$

$$= \int_0^{\pi} (\pi-x) \sin^3(\pi-x) dx \quad \left( \because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$I = \int_0^{\pi} (\pi-x) \sin^3 x dx = \int_0^{\pi} \pi \sin^3 x dx - \int_0^{\pi} x \sin^3 x dx$$

$$= \int_0^{\pi} \pi \sin^3 x dx = I$$

$$\Rightarrow 2I = \int_0^{\pi} \pi \sin^3 x dx = \pi \int_0^{\pi} \frac{3 \sin x - \sin 3x}{4} dx$$

$$= \frac{\pi}{4} \left( -3 \cos x + \frac{\cos 3x}{3} \right)_0^{\pi} = \frac{\pi}{4} \left( -3 \cdot -1 - \frac{1}{3} + 3 - \frac{1}{3} \right)$$

$$= \frac{\pi}{4} (6 - 2/3) = \frac{\pi}{4} \cdot 16/3$$

$$\therefore I = \frac{\pi}{2} \cdot \frac{16}{3} = \frac{2\pi}{3}$$

$$9. \int_0^{\pi} \frac{x}{1+\sin x} dx$$

$$\text{Sol. } I = \int_0^{\pi} \frac{x}{1+\sin x} dx \quad \dots (i)$$

$$I = \int_0^{\pi} \frac{(\pi-x) dx}{1+\sin(\pi-x)} = \int_0^{\pi} \frac{\pi dx}{1+\sin x} - \int_0^{\pi} \frac{x dx}{1+\sin x}$$

$$= \int_0^{\pi} \frac{\pi dx}{1+\sin x} - I$$



$$\begin{aligned}
2I &= \int_0^{\pi} \frac{\pi dx}{1 + \sin x} \Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \frac{dx}{1 + \sin x} \\
&= \frac{\pi}{2} \int_0^{\pi} \frac{(1 - \sin x)}{1 - \sin^2 x} dx = \frac{\pi}{2} \int_0^{\pi} \left( \frac{1 - \sin x}{\cos^2 x} \right) dx \\
&= \frac{\pi}{2} \left( \int_0^{\pi} \frac{1}{\cos^2 x} - \int_0^{\pi} \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx \right) \\
&= \frac{\pi}{2} \int_0^{\pi} \sec^2 x dx - \int_0^{\pi} \sec x \cdot \tan x dx \\
&= \frac{\pi}{2} \left( [\tan x]_0^{\pi} - [\sec x]_0^{\pi} \right) \\
&= \frac{\pi}{2} [(0 - 0) - (-1 - 1)] = \frac{\pi}{2} \cdot 2 = \pi
\end{aligned}$$

10.  $\int_0^{\pi} \frac{x \sin^3 x}{1 + \cos^2 x} dx$

Sol.  $I = \int_0^{\pi} \frac{x \sin^3 x}{1 + \cos^2 x} dx$

$$\begin{aligned}
&= \int_0^{\pi} \frac{(\pi - x) \sin^3(\pi - x)}{1 + \cos^2(\pi - x)} dx \\
&= \int_0^{\pi} \frac{(\pi - x) \sin^3 x}{1 + \cos^2 x} dx \\
&= \pi \int_0^{\pi} \frac{\sin^3 x}{1 + \cos^2 x} dx - \int_0^{\pi} \frac{x \sin^3 x}{1 + \cos^2 x} dx \\
&= \pi \int_0^{\pi} \frac{\sin^3 x}{1 + \cos^2 x} dx - I \\
2I &= \int_0^{\pi} \frac{\sin^3 x dx}{1 + \cos^2 x}
\end{aligned}$$

Put  $t = \cos x \Rightarrow dt = -\sin x dx$

$$2I = \int_1^{-1} \frac{(1-t^2)}{1+t^2} dt = \int_{-1}^1 \frac{1-t^2}{1+t^2} dt$$

$$\begin{aligned}
&= \int_{-1}^1 \left( -1 + \frac{2}{1+t^2} \right) dt = \left[ -t + 2 \tan^{-1} t \right]_{-1}^1 \\
&= \left[ -1 + 2 \tan^{-1} 1 \right] - \left[ -1 + 2 \tan^{-1}(-1) \right] \\
&= -1 + 2 \cdot \frac{\pi}{4} + 1 - 2 \left( -\frac{\pi}{4} \right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi
\end{aligned}$$

$$I = \frac{\pi}{2}$$

11.  $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Sol.  $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Put  $x = \tan \theta$

$$dx = \sec^2 \theta d\theta$$

$$x = 0 \Rightarrow \theta = 0$$

$$x = 1 \Rightarrow \theta = \pi/4$$

$$\begin{aligned}
I &= \int_0^1 \frac{\log(1+x)}{1+x^2} dx = \int_0^{\pi/4} \frac{\log(1+\tan \theta) \sec^2 \theta d\theta}{(1+\tan^2 \theta)} \\
&= \int_0^{\pi/4} \log(1+\tan \theta) d\theta
\end{aligned}$$

$$\text{let } I = \int_0^{\pi/4} \log(1+\tan \theta) d\theta$$

$$= \int_0^{\pi/4} \log \left[ 1 + \tan \left( \frac{\pi}{4} - \theta \right) \right] d\theta$$

$$= \int_0^{\pi/4} \log \left[ 1 + \frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \tan \theta} \right] d\theta$$

$$= \int_0^{\pi/4} \log \left[ 1 + \frac{1 - \tan \theta}{1 + \tan \theta} \right] d\theta$$

$$= \int_0^{\pi/4} \log \left[ \frac{1 + \tan \theta + 1 - \tan \theta}{1 + \tan \theta} \right] d\theta$$

$$\begin{aligned}
&= \int_0^{\pi/4} \log \frac{2}{1 + \tan \theta} d\theta \\
&= \int_0^{\pi/4} [\log 2 - \log(1 + \tan \theta)] d\theta \\
&= \log 2 \int_0^{\pi/4} d\theta - \int_0^{\pi/4} \log(1 + \tan \theta) d\theta \\
&= \log 2 \int_0^{\pi/4} d\theta - I \\
2I &= \log 2 (\theta)_0^{\pi/4} = (\log 2) \frac{\pi}{4}
\end{aligned}$$

$$\therefore I = \frac{\pi}{8} \log 2$$

**12.**  $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$

**Sol.**  $I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x) \sin(\pi - x) dx}{1 + \cos^2(\pi - x)}$

$$\begin{aligned}
&= \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x} - \int_0^{\pi} \frac{x \sin x dx}{1 + \cos^2 x} \\
&= \pi \left\{ \tan^{-1}(-\cos x) \right\}_0^{\pi} - I
\end{aligned}$$

$$2I = \pi \left\{ \tan^{-1} 1 - \tan^{-1}(-1) \right\} = \pi \left( \frac{\pi}{4} + \frac{\pi}{4} \right) = 2 \frac{\pi^2}{4}$$

$$I = \frac{\pi^2}{4} \Rightarrow \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}$$

**13.**  $\int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx$

**Sol.**  $I = \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx \text{ ----1.}$

$$\begin{aligned}
&= \int_0^{\pi/2} \frac{\sin^2 \left( \frac{\pi}{2} - x \right)}{\cos \left( \frac{\pi}{2} - x \right) + \sin \left( \frac{\pi}{2} - x \right)} dx
\end{aligned}$$

$$= \int_0^{\pi/2} \frac{\cos^2 x dx}{\sin x + \cos x} \text{-----2.}$$

**Adding 1. and 2.**

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx$$

$$\Rightarrow I = \frac{1}{2} \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

**Consider**  $\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

**Put  $\tan(x/2) = t$**

$$dx = \frac{2dt}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}$$

$$\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \int_0^1 \frac{2tdt}{2t+(1-t^2)}$$

$$= 2 \int_0^1 \frac{dt}{(\sqrt{2})^2 - (t-1)^2} = 2 \cdot \frac{1}{2\sqrt{2}} \left[ \log \frac{\sqrt{2}+t-1}{\sqrt{2}-t+1} \right]_0^1$$

$$= \frac{1}{\sqrt{2}} \left( \log 1 - \log \frac{\sqrt{2}-1}{\sqrt{2}+1} \right)$$

$$= \frac{1}{\sqrt{2}} \log \frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$= \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)^2 = \frac{2}{\sqrt{2}} \log(\sqrt{2}+1)$$

$$I = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$$

**14. Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous periodic function and  $T$  is the period of it. Let a**

**$\in \mathbb{R}$ . Then prove that for any positive integer  $n$ ,  $\int_0^{a+nT} f(x)dx = n \int_0^{a+T} f(x)dx$ .**

**Sol.**  $\int_0^{a+nT} f(x)dx = \int_0^{a+T} f(x)dx + \int_{a+T}^{a+2T} f(x)dx \dots + \int_{a+(r-1)T}^{a+rT} f(x)dx + \dots + \int_{a+(n-1)T}^{a+nT} f(x)dx \dots(1)$

**Consider ( r+1)th integral of RHS**

$$\int_{a+rT}^{a+(r+1)T} f(x)dx$$

**Let  $x = y + rT \Rightarrow dx = dy$**

$$\mathbf{x = a + rT \Rightarrow y = a}$$

$$\mathbf{x = a + (r + 1)T \Rightarrow y = a + T}$$

$$\int_{a+rT}^{a+(r+1)T} f(x)dx = \int_a^{a+T} f(y+rT)dy$$

$$= \int_a^{a+T} f(y)dy \quad (\mathbf{f \text{ is periodic}})$$

$$= \int_a^{a+T} f(x)dx$$

Similarly we can show that each integral of (1) is equal to  $\int_a^{a+T} f(x)dx$ .

$$\therefore \int_0^{a+nT} f(x)dx = \int_0^{a+T} f(x)dx + \int_0^{a+T} f(x)dx \dots n \text{ terms} = n \int_0^{a+T} f(x)dx$$

## REDUCTION FORMULAE

### THEOREM 1 :

If  $I_n = \int_0^{\pi/2} \sin^n x \, dx$  then  $I_n = \frac{n-1}{n} I_{n-2}$ .

**Proof :**

$$I_n = \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \sin^{n-1} x \cdot \sin x \, dx$$

$$= \left[ -\sin^{n-1} x \cos x \right]_0^{\pi/2} + \int_0^{\pi/2} (n-1) \sin^{n-2} x \cdot \cos^2 x \, dx$$

$$= (n-1) \int_0^{\pi/2} \sin^{n-2} x (1 - \sin^2 x) \, dx$$

$$= (n-1) \left[ \int_0^{\pi/2} \sin^{n-2} x \, dx - \int_0^{\pi/2} \sin^n x \, dx \right]$$

$$= (n-1)I_{n-2} - (n-1)I_n$$

$$I_n (1 + n - 1) = (n-1)I_{n-2} \Rightarrow I_n n = (n-1)I_{n-2}$$

$$\therefore I_n = \frac{n-1}{n} I_{n-2} \text{ -----(1)}$$

## Note

**In (1), replace n by n-2, n-3, ----- then**

$$I_n = \frac{n-1}{n} I_{n-2} \Rightarrow I_{n-2} = \frac{n-3}{n-2} I_{n-4} \Rightarrow I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \dots \cdot I_0 \text{ or } I_1 \text{ according as n is even or odd.}$$

$$\text{But } I_0 = \int_0^{\pi/2} \sin^0 x \, dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$$\begin{aligned} I_1 &= \int_0^{\pi/2} \sin x \, dx = [-\cos x]_0^{\pi/2} \\ &= -\cos \frac{\pi}{2} + \cos 0 = -0 + 1 = 1 \end{aligned}$$

$$\therefore \int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \dots \cdot \frac{1}{2} \cdot \frac{\pi}{2} \text{ if n is even.}$$

$$\therefore \int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \dots \cdot \frac{2}{3} \cdot 1 \text{ if n is odd.}$$

## THEOREM 2 :

$$\text{If } I_n = \int_0^{\pi/2} \cos^n x \, dx \text{ then } I_n = \frac{n-1}{n} I_{n-2}.$$

$$I_n = \int_0^{\pi/2} \cos^n x \, dx = \int_0^{\pi/2} \cos^n \left( \frac{\pi}{2} - x \right) dx = \int_0^{\pi/2} \sin^n x \, dx$$

$$\text{THEOREM 3 : If } I_n = \int_0^{\pi/4} \tan^n x \, dx \text{ then } I_n + I_{n-2} = \frac{1}{n-1}.$$

## Proof :

$$\begin{aligned} I_n &= \int_0^{\pi/4} \tan^n x \, dx = \int_0^{\pi/4} \tan^{n-2} x \tan^2 x \, dx \\ &= \int_0^{\pi/4} \tan^{n-2} x (\sec^2 x - 1) dx = \int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\pi/4} \tan^{n-2} x \, dx \end{aligned}$$

$$= \left[ \frac{\tan^{n-1} x}{n-1} \right]_0^{\pi/4} - I_{n-2} = \frac{1}{n-1} - I_{n-2}$$

$$\therefore I_n + I_{n-2} = \frac{1}{n-1}$$

**THEOREM 4 :**

If  $I_n = \int_0^{\pi/4} \sec^n x \, dx$  then  $I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$ .

**Proof :**

$$\begin{aligned}
 I_n &= \int_0^{\pi/4} \sec^n x \, dx = \int_0^{\pi/4} \sec^{n-2} x \sec^2 x \, dx \\
 &= \left[ \sec^{n-2} x \tan x \right]_0^{\pi/4} - \int_0^{\pi/4} (n-2) \sec^{n-2} x \sec x \tan^2 x \, dx \\
 &= (\sqrt{2})^{n-2} - (n-2) \int_0^{\pi/4} \sec^{n-2} x (\sec^2 x - 1) \, dx \\
 &= (\sqrt{2})^{n-2} - (n-2) \left[ \int_0^{\pi/4} \sec^n x \, dx - \int_0^{\pi/4} \sec^{n-2} x \, dx \right] \\
 &= (\sqrt{2})^{n-2} - (n-2) I_n + (n-2) I_{n-2} \\
 I_n (1+n-2) &= (\sqrt{2})^{n-2} + (n-2) I_{n-2} \\
 \Rightarrow I_n (n-1) &= (\sqrt{2})^{n-2} + (n-2) I_{n-2} \\
 \therefore I_n &= \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}
 \end{aligned}$$

**THEOREM 5:** If  $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$  then  $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$ .

**Proof :**

$$\begin{aligned}
 I_{m,n} &= \int_0^{\pi/2} \sin^m x \cos^n x \, dx = \int_0^{\pi/2} \sin^{m-1} (\sin x \cos^n x) \, dx \\
 &= \left[ \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} \right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos^{n+1} x}{n+1} (m-1) \sin^{m-2} x \cos x \, dx \\
 &= \frac{m-1}{n+1} \int_0^{\pi/2} \sin^{m-2} x \cos^n x \cos^2 x \, dx \\
 &= \frac{m-1}{n+1} \int_0^{\pi/2} \sin^{m-2} x \cos^n x (1 - \sin^2 x) \, dx
 \end{aligned}$$

$$= \frac{m-1}{n+1} \int_0^{\pi/2} \sin^{m-2} x \cos^n x - \sin^m x \cos^n x dx$$

$$= \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n}$$

$$\Rightarrow I_{m,n} \left( 1 + \frac{m-1}{n+1} \right) = \frac{m-1}{n+1} I_{m-2,n}$$

$$\Rightarrow I_{m,n} \left( \frac{n+m}{n+1} \right) = \frac{m-1}{n+1} I_{m-2,n}$$

$$\therefore I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \text{ -----(1)}$$

**Note: replacing m by m-2, m-4,**

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n} = \frac{m-1}{m+n} \frac{m-3}{m+n-2} I_{m-4,n} = \frac{m-1}{m+n} \frac{m-3}{m+n-2} \frac{m-5}{m+n-4} \dots I_{0,n}$$

**or  $I_{1,n}$  according as n is even or odd.**

$$\text{But } I_{0,1} = \int_0^{\pi/2} \sin^0 x \cos^n x dx = \int_0^{\pi/2} \cos^n x dx$$

$$I_{1,n} = \int_0^{\pi/2} \sin x \cos^n x dx = \left[ -\frac{\cos^{n+1} x}{n+1} \right]_0^{\pi/2} = \frac{1}{n+1}$$

$$\therefore I_{m,n} = \frac{m-1}{m+n} \frac{m-3}{m+n-2} \dots \frac{1}{n+1} \text{ if m is odd}$$

$$= \frac{m-1}{m+n} \frac{m-3}{m+n-2} \dots \int_0^{\pi/2} \cos^n x dx \text{ if m is even}$$

**COROLLARY 2: If  $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x dx$  then  $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$ .**