ELLIPSE

EXERCISE -- 4 (B)

1. Find the equation of the tangent and normal to the ellipse $x^2 + 8y^2 = 33$ at $(-1, 2)$.

Sol. Given ellipse $S = x^2 + 8y^2 = 33$

Equation of the tangent is $S_1 = 0$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

$$x(-1) + 8y(2) = 33$$

$$\Rightarrow -x + 16y = 33$$

$$\Rightarrow x - 16y + 33 = 0$$

Equation of the normal is $16x + y + k = 0$

It passes through $P(-1, 2)$ $-16 + 2 + k = 0 \Rightarrow k = 14$

Equation of the normal is $16x + y + 14 = 0$.

2. Find the equation of the tangent and normal to the ellipse $x^2+2y^2-4x+12y+14=0$ at $(2, -1)$.

Sol. Given ellipse $S = x^2+2y^2-4x+12y+14=0$

Equation of the tangent is $S_1 = 0$

$xx_1 + 2yy_1 - 2(x + x_1) + 6(y + y_1) + 14 = 0$

$\Rightarrow 2x - 2y - 2(x + 2) + 6(y - 1) + 14 = 0$

$\Rightarrow 4y + 4 = 0$

$y = -1$ required equation of tangent.

Slope of tangent is 0

Equation of normal be $y + 1 = \frac{-1}{0}(x - 2)$

$x = 2$ equation of normal.

3. Find the equation of the tangents to $9x^2 + 16y^2 = 144$ which makes equal intercepts on coordinate axes.

Ans: $x \pm y \pm 5 = 0$

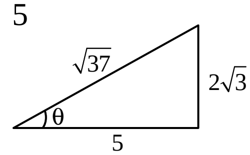
4. Find the coordinates of the points on the ellipse $x^2 + 3y^2 = 37$ at which the normal is parallel to the line $6x - 5y = 2$.

Sol. Equation of the ellipse is $x^2 + 3y^2 = 37 \Rightarrow \frac{x^2}{37} + \frac{y^2}{\left(\frac{37}{3}\right)} = 1$

Slope of the normal = $\frac{a\sin\theta}{b\cos\theta} = \frac{\sqrt{37}\sin\theta}{\sqrt{\frac{37}{3}}\cos\theta} = \sqrt{3}\tan\theta$

The normal is parallel to $6x - 5y = 2$

$\therefore\ \sqrt{3}\tan\theta = \frac{6}{5} \Rightarrow \tan\theta = \frac{6}{5\sqrt{3}} = \frac{2\sqrt{3}}{5}$

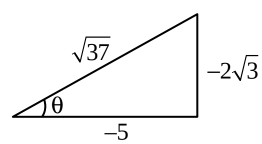


Case I :

The coordinates of P are $(a\cos\theta, b\sin\theta) = \left(\sqrt{37}\frac{5}{\sqrt{37}}, \frac{\sqrt{37}}{\sqrt{3}}\cdot\frac{2\sqrt{3}}{\sqrt{37}}\right) = (5, 2)$

Case II :

The coordinates of P are $(a\cos\theta, b\sin\theta) = \left(\sqrt{37}\frac{(-5)}{\sqrt{37}}, \frac{\sqrt{37}}{\sqrt{3}}\cdot\frac{-2\sqrt{3}}{\sqrt{37}}\right) = (-5, -2)$



5. Find the value of k if $4x + y + k = 0$ is a tangent to the ellipse $x^2 + 3y^2 = 3$.

Sol. Equation of the ellipse is $x^2 + 3y^2 = 3$

$\frac{x^2}{3} + \frac{y^2}{1} = 1$

$\Rightarrow a^2 = 3, b^2 = 1$

Equation of the line is $4x + y + k = 0$

$\Rightarrow y = -4x - k$

$\Rightarrow m = -4, c = -k$

Above line is a tangent to the ellipse

$\Rightarrow c^2 = a^2 m^2 + b^2$

$(-k)^2 = 3(-4)^2 + 1$

$k^2 = 48 + 1 = 49$

$k = \pm 7.$

6. Find the condition for the line xcosα + ysinα = p to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol. Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ …(i)

Equation of the line is $x\cos\alpha + y\sin\alpha = p$

$\Rightarrow y\sin\alpha = -x\cos\alpha + p$

$\Rightarrow y = -x\frac{\cos\alpha}{\sin\alpha}+\frac{p}{\sin\alpha}$

$\therefore\ m = -\frac{\cos\alpha}{\sin\alpha}, c = \frac{p}{\sin\alpha}$

Above line is a tangent to the ellipse

$\Rightarrow c^2 = a^2m^2 + b^2$

$\Rightarrow \frac{p^2}{\sin^2\alpha} = a^2\frac{\cos^2\alpha}{\sin^2\alpha}+b^2$

$\Rightarrow p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha.$

II.

1. Find the equations of tangent and normal to the ellipse $2x^2 + 3y^2 = 11$ at the point whose ordinate is 1.

Sol. Equation of the ellipse is $S = 2x^2 + 3y^2 = 11$

Given y = 1 (I,e., y coordinate is 1)

$\Rightarrow 2x^2 + 3 = 11 \Rightarrow 2x^2 = 8 \Rightarrow x = \pm 2$

Points on the ellipse are P(2, 1) and Q(–2, 1)

Case I P(2, 1)

Equation of the tangent is $S_1 = 0$

$\Rightarrow 2x\cdot 2 + 3y\cdot 1 = 11 \Rightarrow 4x + 3y = 11$

The normal is perpendicular to the tangent.

Equation of the normal at P can be taken as
3x – 4y = k.

The normal passes through P(2, 1)

$6 - 4 = k \Rightarrow k = 2$

Equation of the normal at P is $3x - 4y = 2$.

Case II : Q(–2, 1)

Equation of the tangent at Q is $S_2 = 0$

$\Rightarrow 2x(-2) + 3y.1 = 11$

$\Rightarrow -4x + 3y = 11$

$4x - 3y + 11 = 0$

Equation of the normal can be taken as $3x + 4y = k$

The normal passes through Q(–2, 1)

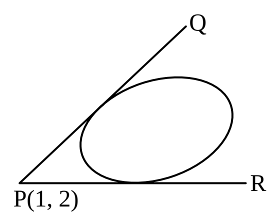
$-6 + 4 = k \Rightarrow k = -2$

Equation of the normal at Q is $3x + 4y = -2$

or $3x + 4y + 2 = 0$.

2. Find the equations to the tangents to the ellipse, $x^2 + 2y^2 = 3$ drawn from the point (1, 2) and also find the angle between these tangents.

Sol. Equations of the ellipse is $x^2 + 2y^2 = 3$



$$\Rightarrow \frac{x^2}{3} + \frac{y^2}{3/2} = 1$$

$\Rightarrow a^2 = 3, b^2 = 3/2$

Let m be the slope of the tangent which is passing through P(1, 2)

Equation of the tangent is

$y - 2 = m(x - 1) = mx - m$

$y = mx + (2 - m)$

Above line is a tangent to the ellipse

$\Rightarrow c^2 = a^2m^2 + b^2$

$$\Rightarrow (2-m)^2 = 3(m^2) + \frac{3}{2}$$

$$\Rightarrow 4 + m^2 - 4m = 3m^2 + \frac{3}{2}$$

$$\Rightarrow 2m^2 + 4m - \frac{5}{2} = 0$$

$$\Rightarrow 4m^2 + 8m - 5 = 0$$

$$\Rightarrow (2m-1)(2m+5) = 0$$

$$m = \frac{1}{2} \text{ or } -\frac{5}{2}$$

Case I : $m = 1/2$

Equation of the tangent is

$$\Rightarrow y = \frac{1}{2}x + 2 - \frac{1}{2} = \frac{x}{2} + \frac{3}{2}$$

$\Rightarrow 2y = x + 3$

$\Rightarrow x – 2y + 3 = 0$

Case II : $m = –5/2$

Equation of the tangent is

$$y = -\frac{5}{2}x + (2 + \frac{5}{2}) = -\frac{5x}{2} + \frac{9}{2}$$

$\Rightarrow 2y = –5x + 9$

$\Rightarrow 5x + 2y – 9 = 0$

Angle between the tangents is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{1}{2} + \frac{5}{2}}{1 + \left(\frac{1}{2}\right)\left(-\frac{5}{2}\right)}\right| = \left|\frac{3}{1 - \frac{5}{4}}\right| = |-12| = 12$$

$\Rightarrow \theta = \tan^{-1} 12.$

3. Find the equation of tangents to the ellipse $2x^2 + y^2 = 8$ which are parallel to $x – 2y + 4 = 0$.

Sol. Equation of the ellipse is $2x^2 + y^2 = 8$

$$\Rightarrow \frac{x^2}{4} + \frac{y^2}{8} = 1$$

Equation of the tangent parallel to $x – 2y + 4 = 0$.

Is $x – 2y + k = 0$.

$$\Rightarrow y = \frac{x}{2} + \frac{k}{2} \quad \Rightarrow \text{ m =1/2 and c =k/2}$$

Above line is a tangent to the ellipse

$\Rightarrow c^2 = a^2m^2 + b^2$

$$\Rightarrow \frac{k^2}{4} = 4.\frac{1}{4} + 8 \Rightarrow k^2 = 36 \Rightarrow k = \pm 6$$

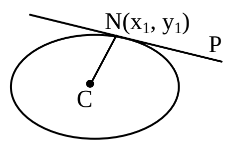
Equation of tangents are

$x - 2y \pm 6 = 0$

III.

1. Show that the feet of the perpendicular drawn from the centre on any tangent to the ellipse lies on the curve $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

Sol. Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Equation of the tangent at P(θ) is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

Slope of the tangent

$$PN = \frac{-\left(\frac{\cos\theta}{a}\right)}{\left(\frac{\sin\theta}{b}\right)} = -\frac{b\cos\theta}{a\sin\theta}$$

Let N (x_1, y_1) be the foot of the perpendicular from C(0, 0) to any tangent.

Slope of CN = $\frac{y_1}{x_1}$.

$\therefore$ Slope of PT $\times$ slope of CN = –1

$$-\frac{b\cos\theta}{a\sin\theta}\cdot\frac{y_1}{x_1} = -1$$

$$\frac{\cos\theta}{ax_1} = \frac{\sin\theta}{by_1} = \frac{1}{\sqrt{a^2x_1^2 + b^2y_1^2}} = k$$

$$\frac{x_1}{a}\cos\theta + \frac{y_1}{b}\sin\theta = 1$$

$$\cos\theta = \frac{ax_1}{k},\ \sin\theta = \frac{by_1}{k}$$

$$\frac{x_1}{a}\cdot ax_1 + \frac{y_1}{b}\cdot by_1 = k$$

$x_1^2 + y_1^2 = k$

$N(x_1, y_1)$ is a point on $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

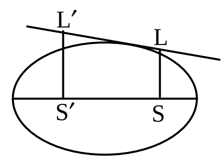
$\Rightarrow \frac{x_1}{a}\cos\theta + \frac{y_1}{b}\sin\theta = 1$

$\Rightarrow x_1^2 + y_1^2 = \sqrt{a^2x_1^2 + b^2y_1^2}$ (or)

$\left(x_1^2 + y_1^2\right)^2 = a^2x_1^2 + b^2y_1^2$

Locus of $N(x_1, y_1)$ is $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$

2. Show that the locus of the feet of the perpendiculars drawn from foci on any tangent of the ellipse is the auxiliary circle.



Sol. Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of the tangent to the ellipse is

$y = mx \pm \sqrt{a^2m^2 + b^2}$

$\Rightarrow y - mx = \pm\sqrt{a^2m^2 + b^2}$ …(1)

Equation to the perpendicular from either focus (±ae, 0) on this tangent is

$y = -\frac{1}{m}(x \pm ae) \Rightarrow my = -(x \pm ae)$

$\Rightarrow my + x = \pm ae$ …(2)

Squaring and adding (1) and (2)

$(y - mx)^2 + (my + x)^2 = a^2m^2 + b^2 + a^2e^2$

$\Rightarrow y^2 + m^2x^2 - 2mxy + m^2y^2 + x^2 + 2mxy = a^2m^2 + a^2 - a^2e^2 + a^2e^2$

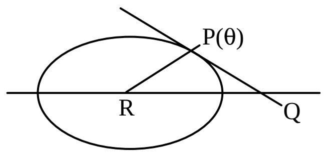
$\Rightarrow (x^2 + y^2)(1 + m^2) = a^2(1 + m^2)$

$\Rightarrow x^2 + y^2 = a^2$

The locus is the auxiliary circle concentric with the ellipse.

3. The tangent and normal to the ellipse $x^2 + 4y^2 = 4$ at a point $P(\theta)$ meets the major axis at Q and R respectively. If $0 < \theta < \pi/2$ and QR = 2, then show that $\theta = \cos^{-1}(2/3)$.

Sol.



Equation of the ellipse is $x^2 + 4y^2 = 4$

$$\frac{x^2}{4}+\frac{y^2}{1}=1$$

Equation of the tangent at $P(\theta)$ is

$$\frac{x}{2}\cdot\cos\theta+\frac{y}{1}\sin\theta=1$$

Equation of x-axis (i.e., major axis) is y = 0

$$\frac{x}{2}\cdot\cos\theta=1 \Rightarrow x=\frac{2}{\cos\theta}$$

Coordinates of Q are $\left(\frac{2}{\cos\theta},0\right)$

Equation of the normal at $P(\theta)$ is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \Rightarrow \frac{2x}{\cos\theta}-\frac{y}{\sin\theta}=3$$

Substituting y = 0 we get $\frac{2x}{\cos\theta}=3$

$$\Rightarrow x=\frac{3}{2}\cos\theta$$

Coordinates of R are $\left(\frac{3}{2}\cos\theta,0\right)$

$$QR=\left(-\frac{3}{2}\cos\theta+\frac{2}{\cos\theta}\right)=\frac{-3\cos^2\theta+4}{2\cos\theta}$$

Given QR = 2

$$\Rightarrow \frac{-3\cos^2\theta+4}{2\cos\theta}=2$$

$$\Rightarrow -3\cos^2\theta+4=4\cos\theta$$

$$\Rightarrow 3\cos^2\theta + 4\cos\theta - 4 = 0$$

$$\Rightarrow (3\cos\theta - 2)(\cos\theta + 2) = 0$$

$\Rightarrow 3\cos\theta - 2 = 0 \Rightarrow \cos\theta = 2/3$

$\Rightarrow \cos\theta + 2 = 0 \Rightarrow \cos\theta = -2$

$\Rightarrow \cos\theta = 2/3$ or -2

$$\Rightarrow \cos\theta = \frac{2}{3}$$

i.e. $\theta = \cos^{-1}\left(\frac{2}{3}\right)$.

THEOREM

The equation of the polar of the point $P(x_1, y_1)$ with respect to the ellipse $S = 0$ is $S_1 = 0$.

Note : If P is an external point of the ellipse S=0, then the polar of P meets the ellipse in two points and the polar becomes the chord of contact of P.

Note : If P lies on the ellipse S = 0, then the polar of P becomes the tangents at P to the ellipse S = 0.

Note 3 : If P is an internal point of the ellipse S=0, then the polar of P does not meet the ellipse S = 0.

THEOREM

The pole of the line lx + my + n = 0 (n ≠ 0) with respect to the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ **is** $\left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right)$**.**

Proof :

Equation of the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

Let $P(x_1, y_1)$ be the pole of the line :

$$lx + my + n = 0 \qquad \ldots(1)$$

The polar of P with respect to the ellipse is $S_1 = 0$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0 \qquad \ldots(2)$$

Now (1) and (2) represent the same line.

$$\therefore \frac{x_1}{a^2l} = \frac{y_1}{b^2m} = \frac{-1}{n} \Rightarrow x_1 = \frac{-a^2l}{n}, y_1 = \frac{-b^2m}{n}$$

$$\therefore \text{Pole } P = \left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right).$$

CONJUGATE POINTS

Two points P and Q are said to be conjugate with respect to the ellipse S = 0 if the polar of P with respect to S = 0 passes through Q.

Note : The condition for the points $P(x_1, y_1)$,$Q(x_2, y_2)$ to be conjugate with respect to the ellipse S = 0 is $S_{12} = 0$.

THEOREM

The condition for the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ to be conjugate with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $a^2l_1l_2 + b^2m_1m_2 = n_1n_2$.

Proof :

Pole of $l_1x + m_1y + n_1 = 0$ with respect to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P\left(\frac{-a^2l_1}{n_1}, \frac{-b^2m_1}{n_1}\right)$.

Given lines are conjugate $\Rightarrow$

P lies on $l_2x + m_2y + n_2 = 0$

$$\Rightarrow l_2\left(\frac{-a^2l_1}{n_1}\right)+m_2\left(\frac{-b^2m_1}{n_1}\right)+n_2=0$$

$$\Rightarrow -a^2l_1l_2 - b^2m_1m_2 + n_1n_2 = 0$$

$$\Rightarrow a^2l_1l_2 + b^2m_1m_2 = n_1n_2$$

THEOREM

The equation to the pair of tangents to the ellipse S = 0 from $P(x_1, y_1)$ is $S_1^2 = S_{11}S$.

THEOREM

The equation of the tangent at $P(\theta)$ on the ellipse S = 0 is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$.

THEOREM

The equation of the normal at $P(\theta)$ on the ellipse S = 0 is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$.

THEOREM

Four normals can be drawn from any point to the ellipse and the sum of the eccentric angles of their feet is an odd multiple of π.

EXERCISE – 4(C)

1. Find the pole of the line 21x – 16y – 12 = 0 w.r.t to the ellipse $3x^2 + 4y^2 = 12$.

Sol. Equation of the ellipse is $3x^2 + 4y^2 = 12$

$$S = \frac{x^2}{4} + \frac{y^2}{3} = 1$$

Pole of the line 21x – 16y – 12 = 0 is $\left(\frac{-a^2\ell}{n}, \frac{-b^2m}{n}\right)$

$$= \left(\frac{-4 \cdot 21}{-12}, \frac{-3(-16)}{-12}\right) = (7, -4)$$

Pole = (7, –4)

2. Show that the focus of an ellipse is the pole of the corresponding directrix.

Sol. Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of one of the the directrix is x = a/e

$$\Rightarrow x - \frac{a}{e} = 0$$

Pole = $\left(\frac{-a^2\ell}{n}, \frac{-b^2m}{n}\right)$

$$= \left(\frac{-a^2 \cdot 1}{(-a/e)}, \frac{-b^2 \cdot 0}{(-a/e)}\right) = (ae, 0)$$

Pole = (ae, 0) is the corresponding focus.

Similarly we can prove the result for the other directrix and the corresponding focus.

3. Find the pole of the line 5x + 7y + 8 = 0 w.r.to $5x^2 + 7y^2 = 8$.

Ans: (–1, –1)

4. Show that (2, –3), (18, 4) are conjugate points w.r.to $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Sol. Equation of the ellipse is S = $\frac{x^2}{9} + \frac{y^2}{4} = 1$

P(2, –3), Q(18, 4) are the given points.

$$\Rightarrow S_{12} = \frac{2.18}{9} + \frac{-3.4}{4} = 4 - 4 = 0$$

Therefore, given points are conjugate points.

II METHOD.

Polar of P w.r.to the ellipse is $\frac{2x}{9}-\frac{3y}{4}=1$

$\frac{2\cdot 18}{9}-\frac{3}{4}\cdot 4=4-3=1$

Polar of P passes through Q

$\therefore$ P and Q are conjugate points w.r.to the ellipse.

5. Find the value of k if the lines 2x + 3y + 1= 0, x + y + k = 0 are conjugate w.r.to the ellipse $3x^2 + 4y^2 = 12$.

Sol. Equation of the ellipse is $3x^2 + 4y^2 = 12$

$$\Rightarrow \frac{x^2}{4}+\frac{y^2}{3}=1$$

$$\Rightarrow a^2 = 4,\ b^2 = 3$$

Equation of the first line is 2x + 3y + 1 = 0

Pole is $\left(\frac{-a^2\ell}{n},\frac{-b^2m}{n}\right)=\left(\frac{-4\cdot 2}{1},\frac{-3\cdot 3}{1}\right)=(-8,-9)$

Since given lines are conjugate, this point lies on x + y + k = 0

$-8 - 9 + k = 0 \Rightarrow k = 17.$

6. Show that the conjugate lines through focus of a ellipse S = 0 are at right angles.

Sol. Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let $l_1x + m_1y + n_1 = 0$, $l_2x + m_2y + n_2 = 0$ be conjugate lines,

$\therefore\ a^2l_1l_2 + b^2m_1m_2 = n_1n_2$ …(1)

The given lines pass through S(ae, 0)

$l_1ae+0+n_1=0 \Rightarrow l_1ae=-n_1$

$l_2ae+0+n_2=0 \Rightarrow l_2ae=-n_2$

$l_1l_2\cdot a^2e^2 = n_1n_2$ …(2)

From (1), (2), we get

$a^2l_1l_2 + b^2m_1m_2 = l_1l_2a^2e^2$

$\Rightarrow l_1l_2(a^2-a^2e^2)+b^2m_1m_2=0$

$\Rightarrow b^2l_1l_2+b^2m_1m_2=0$ $[a^2(1-e^2)=b^2]$

$l_1l_2 + m_1m_2 = 0 \Rightarrow \quad \frac{l_1l_2}{m_1m_2} = -1 \Rightarrow \left(\frac{-l_1}{m_1}\right)\left(\frac{-l_2}{m_2}\right) = -1$

$\therefore$ The given conjugate lines are at right angles.

7. Find the value of k, if (1, 2)(k, –1) are conjugate points with respect to $2x^2 + 3y^2 = 6$.

Sol. Equation of the ellipse is $2x^2 + 3y^2 = 6$

$$\Rightarrow \frac{x^2}{3} + \frac{y^2}{2} = 1$$

Given (1, 2), (k, –1) are conjugate points ,

$$\Rightarrow S_{12} = 0 \Rightarrow \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} = 1$$

$$\Rightarrow \frac{1\cdot k}{3} + \frac{2(-1)}{2} = 1$$

$$\Rightarrow \frac{k}{3} = 1 + 1 = 2$$

$\Rightarrow k = 6$

II.

1. Find the equation of a straight line through the point (2, 1) and conjugate to the straight line 9x + 2y = 1 with respect to the ellipse $3x^2 + 2y^2 = 1$.

Sol. Equation of ellipse is $3x^2 + 2y^2 = 1$

$$\Rightarrow \frac{x^2}{1/3} + \frac{y^2}{1/2} = 1$$

Equation of the given line

$9x + 2y = 1$ i.e., $9x + 2y - 1 = 0$

Pole of the line is $\left(\frac{-a^2\ell}{n}, \frac{-b^2m}{n}\right)$

$$= \left(-\frac{\frac{1}{3}9}{-1}, -\frac{\frac{1}{2}2}{-1}\right) = (3,1)$$

Required line is passing through (3, 1) and also through the point (2, 1).

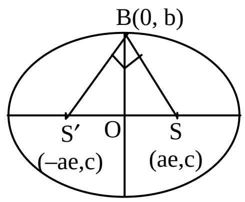
Equation of the line is $y - 1 = \frac{1-1}{3-2}(x-2)$

$y - 1 = 0$

2. An ellipse has OB as minor axis, S, S′ are its foci and the angle. SBS′ is a right angle, then find the eccentricity of the ellipse.

Sol. Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The foci are S(ae, 0) and S′ (–ae, 0)



Coordinates of B are (0, b)

Slope of SB = $\frac{b}{-ae}$, slope of S′B = $\frac{b}{ae}$

SBS′ = 90° $\Rightarrow$ Slope of SB $\times$ slope of S′B = –1

$-\frac{b}{ae}\cdot\frac{b}{ae}=-1$

$b^2=a^2e^2$

$a^2(1-e^2)=a^2e^2$

$1-e^2=e^2 \Rightarrow 2e^2=1$

$e^2=\frac{1}{2}\Rightarrow e=\frac{1}{\sqrt{2}}$

III.

1. Show that the poles of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ w.r.to the circle $x^2 + y^2 = a^2$ lies on the curve $a^2x^2 + b^2y^2 = a^4$.

Sol.

Equation of the circle is $x^2 + y^2 = a^2$

Let P(x_1, y_1) the ploe.

The polar of P with respect to the circle $x^2 + y^2 = a^2$ is $S_1 = 0$

$\Rightarrow xx_1 + yy_1 - a^2 = 0$ -----(i)

(i) is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$c^2 = a^2m^2 + b^2$

$$\Rightarrow \left(\frac{-a^2}{y_1}\right)^2 = a^2 \cdot \left(\frac{-x_1}{y_1}\right)^2 + b^2$$

Places on the curve $a^4 = a^2x^2 + b^2y^2$.

2. **Show that the poles of the tangents to the circle $x^2 + y^2 = a^2 + b^2$ w.r.to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ lies on $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$.**

Sol. Let $P(x_1, y_1)$ be the pole.

Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Equation of the polar is $S_1 = 0 \Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$ ---(i)

Equation of the circle is $x^2 + y^2 = a^2 + b^2$

Centre = (0, 0), $r = \sqrt{a^2+b^2}$

(i) is a tangent to the circle $\Rightarrow$ radius = perpendicular distance from centre to the lilne

$$\Rightarrow \frac{|0+0-1|}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}}} = \sqrt{a^2+b^2}$$

$$\Rightarrow \frac{1}{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}} = a^2+b^2 \Rightarrow \frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}=\frac{1}{a^2+b^2}$$

Locus of $P(x_1, y_1)$ is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$.

PROLEMS FOR PRACTICE

1. **Find the eccentricity, coordinates of foci. Length of latus rectum and equations of directrices of the following ellipses.**

   i) $9x^2 + 16y^2 - 36x + 32y - 92 = 0$

   ii) $3x^2 + y^2 - 6x - 2y - 5 = 0$

2. **Find the equation of the ellipse referred to its major and minor axes as the coordinate axes x, y respectively with latus rectum of length 4 and the distance between foci $4\sqrt{2}$.**

Ans. $x^2 + 2y^2 = 16$

3. **If the length of the latus rectum is equal to half of its minor axis of an ellipse in the standard form, then find the eccentricity of the ellipse.**

Ans. $e = \sqrt{3}/2$

4. **If θ_1, θ_2 are the eccentric angles of the extremities of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b) and e its eccentricity, then show that**

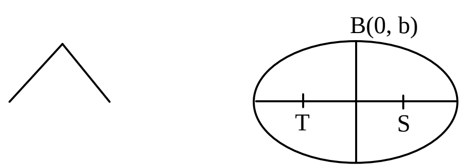
i) $e\cos\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1-\theta_2}{2}$

ii) $\frac{e+1}{e-1}=\cot\left(\frac{\theta_1}{2}\right)\cdot\cot\left(\frac{\theta_2}{2}\right)$

5. **C is the center, AA′ and BB′ are major and minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If PN is the ordinate of a point P on the ellipse then show that $\frac{(PN)^2}{(A'N)(AN)}=\frac{(BC)^2}{(CA)^2}$.**

6. **If S and T are the foci of an ellipse and B is one end of the minor axis. If STB is an equilateral triangle, then find the eccentricity of the ellipse.**

Sol.



Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Foci are S(ae, 0), T(–ae, 0)

B(0, b) is the end of the minor axis

STB is an equilateral triangle

$SB = ST \Rightarrow SB^2 = ST^2$

$\Rightarrow a^2e^2 + b^2 = 4a^2e^2$

$\Rightarrow b^2 = 3a^2e^2$

$\Rightarrow a^2(1 - e^2) = 3a^2e^2$

$1 - e^2 = 3e^2$

$4e^2 = 1 \Rightarrow e^2 = \frac{1}{4}$

Eccentricity of the ellipse : $e=\frac{1}{2}$.

7. Find the equation of the tangent and normal to the ellipse $9x^2 + 16y^2 = 144$ at the end of the latus rectum in the first quadrant.

Sol. Given ellipse is $9x^2 + 16y^2 = 144$

$$\Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$$

$$\Rightarrow e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{16-9}{16}}=\frac{\sqrt{7}}{4}$$

End of the latus rectum in first Quadrant

$$P\left(ae,\frac{b^2}{a}\right)=\left(\sqrt{7},\frac{9}{4}\right)$$

Equation of the tangent at P is $S_1 = 0 \Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

$$x\cdot\frac{\sqrt{7}}{16}+\frac{y}{9}\left(\frac{9}{4}\right)=1$$

$$\frac{\sqrt{7}x}{16}+\frac{y}{4}=1$$

$$\sqrt{7}x+4y=16$$

Equation of the normal at P is

$$\Rightarrow \frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

$$\Rightarrow \frac{16x}{\sqrt{7}}-\frac{9y}{(9/4)}=16-9$$

$$\frac{16x}{\sqrt{7}}-4y=7\Rightarrow 16x-4\sqrt{7}y=7\sqrt{7}$$

8. If a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b) meets its major axis and minor axis in M and N respectively, then prove that $\frac{a^2}{(CM)^2}+\frac{b^2}{(CN)^2}=1$. Where e is the centre of the ellipse.

9. Find the condition for the line

i) lx + my + n = 0 to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

ii) lx + my + n = 0 to be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$**.**

Sol. i) Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Equation of the tangent at P(θ) is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ …(1)

Equation of the given line is lx + my = –n …(2)

(1) and (2) are representing the same line. Therefore,

$$\frac{\cos\theta}{al}=\frac{\sin\theta}{bm}=\frac{1}{-n}$$

$$\Rightarrow \frac{\cos\theta}{al}-\frac{\sin\theta}{bm}=\frac{1}{-n}$$

$$\Rightarrow \cos\theta=-\frac{al}{n}\sin\theta=-\frac{bm}{n}$$

SINCE $\cos^2\theta+\sin^2\theta=1$

$$\Rightarrow \frac{a^2l^2}{n^2}+\frac{b^2m^2}{n^2}=1$$

$\Rightarrow a^2l^2 + b^2m^2 = n^2$ is the required condition.

ii) Let lx + my + n = 0 be normal at P(a)Equation of the normal at P(a) is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \quad \text{…(1)}$$

BUT equation of the normal is Lx + my = –n …(2)

Comparing (1) and (2)

$$\frac{l}{\left(\frac{a}{\cos\theta}\right)}=\frac{m}{\left(\frac{-b}{\sin\theta}\right)}=\frac{n}{a^2-b^2}$$

$$\frac{l\cos\theta}{a}=\frac{-m\sin\theta}{b}=\frac{-n}{a^2-b^2}$$

$$\cos\theta=\frac{-an}{l(a^2-b^2)},\ \sin\theta=\frac{bn}{m(a^2-b^2)}$$

$$\cos^2\theta+\sin^2\theta=1$$

$$\frac{a^2n^2}{l^2(a^2-b^2)^2}+\frac{b^2n^2}{m^2(a^2-b^2)^2}=1$$

$$\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$$ is the required condition.

10. If the normal at one end of a latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through one end of the minor axis, then show that $e^4 + e^2 = 1$.

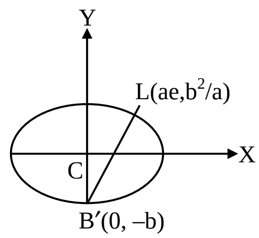
Sol. Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

One end of the latusrectum is $L(ae, b^2/a)$

Equation of the normal at $L(ae, b^2/a)$ is

$$\frac{a^2x}{ae}-\frac{b^2y}{(b^2/a)}=a^2-b^2 \quad \left(\because \frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2\right)$$

$$\frac{ax}{e}-ay=a^2e^2$$



This normal passes through $B'(0, -b)$

$ab = a^2e^2$

$\Rightarrow b = ae^2$

$\Rightarrow b^2 = a^2e^4$

$\Rightarrow a^2(1 - e^2) = a^2e^4$

$\Rightarrow e^4 + e^2 = 1$.

11. If PN is the ordinate of a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the tangent at P meets X - axis at T, then show that $(CN)(CT) = a^2$ where C is the centre of the ellipse.

12. Show that the point of intersection of perpendicular tangents to an ellipse lie on a circle.

Sol. Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let $P(x_1, y_1)$ be the point of intersection of the tangents.

Equation of the tangent is

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

This tangent is passing through $P(x_1, y_1)$

$$y_1 = mx_1 \pm \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow y_1 - mx_1 = \pm\sqrt{a^2m^2 + b^2}$$

$$\Rightarrow (y_1 - mx_1)^2 = a^2m^2 + b^2$$

$$\Rightarrow m^2x_1^2 + y_1^2 - 2mx_1y_1 - a^2m^2 - b^2x$$

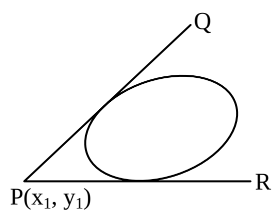
$$\Rightarrow m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - b^2) = 0$$

This is a quadratic equation in m giving two values for m say m_1 and m_2. These are the slopes of the tangents passing through (x_1,y_1).

The tangents are perpendicular $\Rightarrow m_1m_2 = -1$

$$\frac{y_1^2 - b^2}{x_1^2 - a^2} = -1 \Rightarrow y_1^2 - b^2 = -x_1^2 + a^2 \Rightarrow x_1^2 + y_1^2 = a^2 + b^2$$

Locus of $P(x_1, y_1)$ is $x^2 + y^2 = a^2 + b^2$ which is a circle.



This circle is called Director circle of the Ellipse.

13. Find the pole of the line $3x - 5y - 9 = 0$ w.r. to the ellipse $4x^2 + 8y^2 - 16x + 15 = 0$.

Ans. Pole = $\left(\frac{9}{4}, \frac{-5}{24}\right)$

14. Show that the poles of the tangents to the auxiliary circle w.r. to the ellipse lie on the curve $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2}$.

Sol. Let $P(x_1, y_1)$ be the pole.

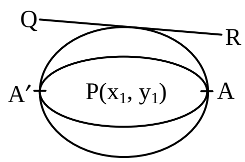
Polar of P w.r. to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \qquad \ldots(1)$$

Equation of the auxiliary circle is $x^2 + y^2 = a^2$

Polar (1) is a tangent to the auxiliary circle.

$$\frac{|0+0-1|}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}}}=a$$



$$\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}=\frac{1}{a^2}$$

Locus of $P(x_1, y_1)$ is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}$.

15. A chord PQ of an ellipse S = 0 subtends a right angle at the centre of the ellipse. Show that the point of intersection of tangents at P and Q lies on another ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$**.**

16. Show that the poles of the tangents of y^2 = 4kx (k > 0) w.r. to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **lies on a parabola.**

Sol. Let $P(x_1, y_1)$ be the pole .

Pplar of P w.r.to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$

$$y=\frac{-b^2x_1}{a^2y_1}\cdot x+\frac{b^2}{y_1}$$

This is a tangent to $y^2 = 4kx \Rightarrow c=\frac{a}{m}$

$$\frac{b^2}{y_1} = \frac{k}{\left(\frac{-b^2x_1}{a^2y_1}\right)} = \frac{-ky_1^2}{-bx_1}$$

$$\Rightarrow y_1^2 = \left(\frac{-b^4}{a^2k}\right)x_1$$

Locus of $P(x_1, y_1)$ is the parabola $y^2 = \left(-\frac{b^4}{a^2k}\right)x$.

17. Show that the poles of normal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ lie on the curve $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$.

Sol.

Equation of the ellipse si $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of the normal at $P(\theta)$ is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = (a^2 - b^2)^2$ … (1).

Let $P(x_1, y_1)$ be the pole of (1)

Polar of P is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ …(2)

Comparing (1) and (2)

$$\frac{\frac{x_1}{a^2}}{\left(\frac{a}{\cos\theta}\right)} = \frac{\frac{y_1}{b^2}}{\left(-\frac{b}{\sin\theta}\right)} = \frac{1}{a^2 - b^2}$$

$$\frac{x_1\cos\theta}{a^3} = \frac{y_1\sin\theta}{-b^3} = \frac{1}{a^2 - b^2}$$

$$(a^2 - b^2)\cos\theta = \frac{a^3}{x_1},\ (a^2 - b^2)\sin\theta = \frac{-b^3}{y_1}$$

$$(a^2 - b^2)^2(\cos^2\theta + \sin^2\theta) = \frac{a^6}{x_1^2} + \frac{b^6}{y_1^2}$$

Locus of $P(x_1, y_1)$ is $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$.