PARABOLA

EXERCISE – 3(B)

1. Find equation of the tangent and normal to the parabola $y^2 = 6x$ at the positive end of the latus rectum.

Sol. Equation of parabola $y^2 = 6x$

$4a = 6 \Rightarrow a = 3/2$

Positive end of the Latus rectum is(a, 2a) = $\left(\frac{3}{2}, 3\right)$

Equation of tangent $yy_1 = 2a(x + x_1)$

$yy_1 = 3(x + x_1)$

$3y = 3\left(x + \frac{3}{2}\right)$

$2y - 2x - 3 = 0$ is the equation of tangent

Slope of tangent is 1

Slope of normal is –1

Equation of normal is $y - 3 = -1\left(x - \frac{3}{2}\right)$

$2x + 2y - 9 = 0$

2. Find the equation of the tangent and normal to the parabola $x^2 - 4x - 8y + 12 = 0$ at (4, 3/2).

Sol.

Equation of the the parabola is

$x^2 - 4x - 8y + 12 = 0$. And point is **(4, 3/2)**

Equation of tangents at (x_1, y_1) is $S_1 = 0$

$4x - 2(x + 4) - 4\left(y + \frac{3}{2}\right) + 12 = 0$

$\Rightarrow 2x - 4y - 2 = 0$

$\Rightarrow x - 2y - 1 = 0$

Equation of normal is$y - y_1 = m(x - x_1)$

m-slope of normal

Slope of tangent is 1/2

Slope of normal is –2. Therefore equation of the normal is

$y - \frac{3}{2} = -2(x - 4) \Rightarrow 2y - 3 = -4x + 16$

$\Rightarrow 4x + 2y - 19 = 0$

3. Find the value of k if the line 2y = 5x + k is a tangent to the parabola $y^2 = 6x$.

Sol.

Equation of the parabola is $y^2 = 6x$

Given line is 2y = 5x + k

$\Rightarrow y = \left(\frac{5}{2}\right)x + \left(\frac{k}{2}\right)$

Therefore $m = \frac{5}{2}$, $c = \frac{k}{2}$

$y = \left(\frac{5}{2}\right)x + \left(\frac{k}{2}\right)$ is a tangent to $y^2 = 6x$

$\Rightarrow c = \frac{a}{m} \Rightarrow \frac{k}{2} = \frac{3/2}{5/2} \Rightarrow k = \frac{6}{5}$

4. Find the equation of the normal to the parabola $y^2 = 4x$ which is parallel to y – 2x + 5 = 0.

Sol. Given the parabola is $y^2 = 4x$

$\therefore$ a = 1

Given line y – 2x + 5 = 0

Slope m = 2

The normal is parallel to the line y – 2x+5 = 0

Slope of the normal = 2

Equation of the normal at 't' is $y + tx = 2at + at^3$

$\therefore$ slope = –t = 2 ($\Rightarrow$t = –2)

Equation of the normal is $y - 2x = 2 \cdot 1(-2) + 1(-2)^3 = -4 - 8 = -12$

2x – y – 12 = 0.

5. Show that the line 2x – y + 2 = 0 is a tangent to the parabola $y^2 = 16x$. Find the point of contact also.

Sol. Given parabola is $y^2 = 16x$

$\Rightarrow$ 4a = 16 $\Rightarrow$ a = 4

Given line is 2x – y + 2 = 0

y = 2x + 2

$\Rightarrow$ m = 2, c = 2

$\frac{a}{m} = \frac{4}{2} = 2 = c$

Therefore given line is a tangent to the parabola.

$\therefore$ Point of contact =

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{4}{2^2}, \frac{2(4)}{2}\right) = (1, 4)$$

6. Find the equation of tangent to the parabola $y^2 = 16x$ inclined at an angle 60° with its axis and also find the point of contact.

Sol.

Given parabola $y^2 = 16x$

Inclination of the tangent is

$\theta = 60° \Rightarrow \quad m = \tan 60° = \sqrt{3}$

Therefore equation of the tangent is $y = mx + \frac{a}{m}$

$$\Rightarrow \; y = \sqrt{3}x + \frac{4}{\sqrt{3}}$$

$$\Rightarrow \; \sqrt{3}y = 3x + 4$$

Point of contact = $\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{4}{3}, \frac{8}{\sqrt{3}}\right)$

II.

1. Find the equations of tangents to the parabola $y^2 = 16x$ which are parallel and perpendicular respectively to the line 2x – y + 5 = 0. Find the coordinates of the points of contact also.

Sol.

Given parabola is $y^2 = 16x$

$\Rightarrow 4a = 16 \Rightarrow a = 4$

Equation of the tangent parallel to $2x - y + 5 = 0$ is $y = 2x + c$

Condition for tangency is $c = \frac{a}{m} = \frac{4}{2} = 2$

Equation of the tangent is $y = 2x + 2 \Rightarrow 2x - y + 2 = 0$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{4}{4}, \frac{8}{2}\right) = (1, 4)$

Equation of the tangent perpendicular to $2x - y + 5 = 0$ is $x+2y+c=0$

$$\Rightarrow 2y = -x - c \Rightarrow y = -\frac{1}{2}x - \frac{1}{2}c$$

If above line is a tangent the $c = a/m$

$$\Rightarrow -\frac{1}{2}c = \frac{4}{\left(-\frac{1}{2}\right)} \Rightarrow c = 16$$

Equation of the perpendicular tangent is

$$y = -\frac{1}{2}x - 8 \Rightarrow 2y = -x - 16$$

$$\Rightarrow x + 2y + 16 = 0$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$$= \left(\frac{4}{(1/4)}, \frac{8}{(-1/2)}\right) = (16, -16).$$

2. If lx + my + n = 0 is a normal to the parabola $y^2 = 4ax$, then show that $al^3 + 2alm^2 + nm^2 = 0$.

Sol. Given parabola is $y^2 = 4ax$

Equation of the normal is $y + tx = 2at + at^3$

$\Rightarrow tx + y - (2at + at^3) = 0$ …(1)

Equation of the given line is

$lx + my + n = 0$ …(2)

(1), (2) are representing the same line, therefore

$$\frac{t}{\ell} = \frac{1}{m} = \frac{-(2at + at^3)}{n}$$

$$\Rightarrow \frac{t}{\ell} = \frac{1}{m} \Rightarrow t = \frac{\ell}{m}$$

$$\Rightarrow \frac{1}{m} = -\frac{(2at + at^3)}{n}$$

$$\Rightarrow \frac{-n}{m} = 2a \cdot t + at^3$$

$$\Rightarrow 2a \cdot \frac{\ell}{m} + a \cdot \left(\frac{\ell}{m}\right)^3 = \frac{2a\ell}{m} + \frac{a\ell^3}{m^3}$$

$\Rightarrow -nm^2 = 2al\,m^2 + al^3$

$\Rightarrow al^3 + 2alm^2 + nm^2 = 0$

3. Show that the equation of common tangents to the circle $x^2 + y^2 = 2a^2$ and the parabola $y^2 = 8ax$ are $y = \pm(x + 2a)$.

Sol.

Given parabola $y^2 = 8ax \Rightarrow y^2 = 4.2ax$

The equation of tangent to parabola is $y = mx + \frac{2a}{m}$.

$m^2x - my + 2a = 0$ …(1)

If (1) is a tangent to the circle $x^2 + y^2 = 2a^2$, then the length of perpendicular from its centre (0, 0) to (1) is equal to the radius of the circle.

$$\left|\frac{2a}{\sqrt{m^2 + m^4}}\right| = a\sqrt{2}$$

$$\Rightarrow 4 = 2(m^4 + m^2)$$

$\Rightarrow m^4 + m^2 - 2 = 0$

$\Rightarrow (m^2 + 2)(m^2 - 1) = 0$ or $m = \pm 1$

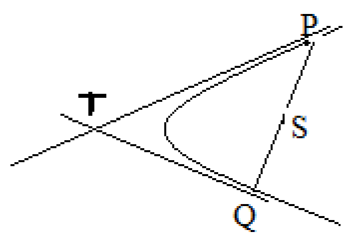
Required tangents are

$$y = (1)x + \frac{2a}{(1)},\ y = (-1)x + \frac{2a}{(-1)}$$

$$\Rightarrow y = \pm(x + 2a)$$

4. Prove that the tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix.

Sol. Let the parabola be $y^2 = 4ax$



Equation of the tangent at $P(t_1)$ is

$$t_1 y = x + at_1^2$$

Equation of the tangent at $Q(t_2)$ is $t_2 y = x + at_2^2$

Solving, point of intersection is $T[at_1 t_2, a(t_1 + t_2)]$

Equation of the chord PQ is $(t_1 + t_2)y = 2x + 2at_1 t_2$

Since PQ is a focal chord, S (a,0) is a point on PQ.

Therefore, $0 = 2a + 2a\,t_1 t_2$

$\Rightarrow \quad t_1 t_2 = -1.$

Therefore point of intersection of the tangents is $[-a, a(t_1 + t_2)]$.

The x coordinate of this point is a constant. And that is $x = -a$ which is the equation of the directrix of the parabola.

Hence tangents are intersecting on the directrix.

5. Find the condition for the line y = mx + c to be a tangent to x^2 = 4ay.

Sol.

Equation of the parabola is $x^2 = 4ay$.----(1)

Equation of the line is $y = mx + c$ ----(2)

Solving above equations,

$x^2 = 4a(mx + c) \Rightarrow x^2 - 4amx - 4ac = 0$ which is a quadratic in x.

If the given line is a tangent to the parabola, the roots of above equation are real and equal.

$\Rightarrow$ b^2 -4ac = 0 $\Rightarrow$ $16a^2m^2 + 16ac = 0$

$\Rightarrow am^2 + c = 0$ $\Rightarrow c = -\ am^2$ is the required condition.

6. **Three normals are drawn (k, 0) to the parabola $y^2 = 8x$ one of the normal is the axis and the remaining two normals are perpendicular to each other, then find the value of k.**

Sol. Equation of parabola is $y^2 = 8x$

Equation of the normal to the parabola is $y + xt = 2at + at^3$ which is a cubic equation in t. therefore it has 3 roots. Say t_1, t_2 ,t_3 . where - t_1, -t_2 ,-t_3 are the slopes of the normals.

This normal is passing through (k, 0)

$\therefore$ $kt = 2at + at^3$

$at^3 + (2a - k)t = 0$

$at^2 + (2a - k) = 0$

Given one normal is axis i.e., x axis and the remaining two are perpendicular. Thererfore

$m_1 = 0 = t_1$, and $m_2m_3 = -1$

$(-t_2)(-t_3) = -1$, $t_2t_3 = -1$

$$\frac{2a-k}{a} = -1 \Rightarrow 2a - k = -a$$

$\Rightarrow k = 2a + a = 3a$

Equation of the parabola is $y^2 = 8x$

$4a = 8 \Rightarrow a = 2$

$k = 3a = 3 \times 2 = 6.$

III.

1. **If the normal at the point t_1 on the parabola $y^2 = 4ax$ meets it again at point t_2 then prove that $t_1t_2 + t_1^2 + 2 = 0$.**

Sol.

Equation of the parabola is $y^2 = 4ax$

Equation of normal at $t_1 = (at_1^2, 2at_1)$ is

$y + xt_1 = 2at_1 + at_1^3$.

This normal meets the parabola again at $(at_2^2, 2at_2)$.

Therefore, $2at_2 + at_2^2t_1 = 2at_1 + at_1^3$

$$\Rightarrow 2\left(t_2 - t_1\right) = t_1\left(t_1^2 - t_2^2\right)$$

$$\Rightarrow 2 = -t_1\left(t_1 - t_2\right)$$

$$\Rightarrow t_1t_2 + t_1^2 + 2 = 0$$

2. **From an external point P tangents are drawn to the parabola $y^2 = 4ax$ and these tangents make angles θ_1, θ_2 with its axis such that $\cot\theta_1 + \cot\theta_2$ is a constant 'd' show that P lies on a horizontal line.**

Sol.

Equation of the parabola is $y^2 = 4ax$

Equation of any tangent to the parabola is $y = mx + \frac{a}{m}$

This tangent passes through $P(x_1, y_1)$, then $y_1 = mx_1 + \frac{a}{m}$

$\Rightarrow my_1 = m^2x_1 + a \Rightarrow m^2x_1 - my_1 + a = 0$

let m_1, m_2 be the roots of the equation

$m_1 + m_2 = \frac{y_1}{x_1}, \ m_1m_2 = \frac{a}{x_1}$ where m_1 and m_2 are the slopes of the tangents.

$\Rightarrow m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

Given $\cot\theta_1 + \cot\theta_2 = d$

$$\Rightarrow \frac{1}{\tan\theta_1} + \frac{1}{\tan\theta_2} = d$$

$$\frac{1}{m_1} + \frac{1}{m_2} = a \Rightarrow \frac{m_1 + m_2}{m_1m_2} = d$$

$$\Rightarrow m_1 + m_2 = d \cdot m_1m_2$$

$$\frac{y_1}{x_1} = d \cdot \frac{a}{x_1} \Rightarrow y_1 = ad$$

Locus of $P(x_1, y_1)$ is $y = ad$ which is a horizontal line.

3. **Show that the common tangent to the circle $2x^2 + 2y^2 = a^2$ and the parabola $y^2 = 4ax$ intersect at the focus of the parabola $y^2 = -4ax$.**

Sol.

Given parabola is $y^2 = 4ax$

Let $y = mx + \frac{a}{m}$ be the tangent. But this is also the tangent to $2x^2 + 2y^2 = a^2$

$\Rightarrow$ Perpendicular distance from centre (0, 0) to the line = radius

$$\Rightarrow \left|\frac{a/m}{\sqrt{m^2+1}}\right| = \frac{a}{\sqrt{2}} \Rightarrow \frac{a^2/m^2}{m^2+1} = \frac{a^2}{2}$$

$$\Rightarrow \frac{2a^2}{m^2} = a^2(m^2+1)$$

$$\Rightarrow 2 = m^4 + m^2 \Rightarrow m^4 + m^2 - 2 = 0$$

$\Rightarrow (m^2 - 1)(m^2 + 2) = 0 \ \ (\because m^2 + 2 \neq 0)$

$m^2 - 1 = 0 \Rightarrow m = \pm 1$

Therefore, equations of the tangents are

y = -x –a and y = x + a .

The point of intersection of these two tangents is (- a, 0) which is the focus of the parabola $y^2 = -4ax$.

POLE AND POLAR

THEOREM

The equation of the polar of the point $P(x_1, y_1)$ with respect to the parabola S = 0 is $S_1 = 0$.

Note . If P is an external point of the parabola S = 0, then the polar of P meets the parabola in two points and the polar becomes the chord of contact of P.

Note .

If P lies on the parabola S = 0, then the polar of P becomes the tangent at P to the parabola S = 0.

Note . If P is an internal point of the parabola S = 0, then the polar of P does not meet the parabola.

THEOREM

The pole of the line lx + my + n = 0 (l ≠ 0) with respect to the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, \frac{-2am}{l}\right)$.

Proof : Equation of the parabola is $y^2 = 4ax$

Equation of the line is lx + my + n = 0 …(1)

Let $P(x_1, y_1)$ be the pole

The polar of P with respect to the parabola is $S_1 = 0$

$yy_1 = 2a(x + x_1)$

$\Rightarrow \ 2ax - y_1y + 2ax_1 = 0$ …(2)

Now (1) and (2) represent the same line.

$\therefore \ \frac{2a}{l} = \frac{-y_1}{m} = \frac{2ax_1}{n} \Rightarrow x_1 = \frac{n}{l}, y_1 = -\frac{2am}{l}$

$\therefore$ Pole P = $\left(\frac{n}{l}, \frac{-2am}{l}\right)$.

Note.

The pole of the line lx + my + n = 0 (m ≠ 0) with respect to the parabola $x^2 = 4ay$ is $\left(\frac{-2al}{m}, \frac{n}{m}\right)$.

CONJUGATE POINTS

Note : The condition for the points $P(x_1, y_1)$,
$Q(x_2, y_2)$ to be conjugate with respect to the parabola $S = 0$ is $S_{12} = 0$.

CONJUGATE LINES

Two lines $L_1 = 0$, $L_2 = 0$ are said to be conjugate with respect to the parabola $S = 0$ if the pole of $L_1 = 0$ lies on $L_2 = 0$.

THEOREM

The condition for the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ to be conjugate with respect to the parabola $y^2 = 4ax$ is $l_1n_2 + l_2n_1 = 2am_1m_2$.

Proof :

Equation of the parabola is $y^2 = 4ax$

Pole of the line $l_1x + m_1y + n_1 = 0$ with respect to $y^2 = 4ax$ is $P\left(\frac{n_1}{l_1}, \frac{-2am_1}{l_1}\right)$.

Given lines are conjugate

$\Rightarrow$ P lies on $l_2x + m_2y + n_2 = 0$.

$$\Rightarrow l_2\left(\frac{n_1}{l_1}\right) + m_2\left(\frac{-2am_1}{l_1}\right) + n_2 = 0$$

$$\Rightarrow l_2n_1 - 2am_1m_2 + l_1n_2 = 0$$

$$\Rightarrow l_1n_2 + l_2n_1 = 2am_1m_2$$

MIDPOINT OF A CHORD

THEOREM

The equation of the chord of the parabola $S = 0$ having $P(x_1, y_1)$ as its midpoint is $S_1 = S_{11}$.

PAIR OF TANGENTS

THEOREM

The equation to the pair of tangents to the parabola $S = 0$ from $P(x_1, y_1)$ is $S_1^2 = S_{11}S$.

EXERCISE – 3(C)

1. Find the pole of the line 2x + 3y + 4 = 0 with respect to the parabola y^2 = 8x.

Sol. Equation of the parabola is $y^2 = 8x$

$4a = 8 \Rightarrow a = 2$

Equation of the given line is $2x + 3y + 4 = 0$

$\ell = 2, m = 3, n = 4$

Pole = $\left(\frac{n}{\ell}, -\frac{2am}{\ell}\right) = \left(\frac{4}{2}, \frac{-2\cdot 2\cdot 3}{2}\right) = (2, -6)$

2. Find the pole of 2x – y – 4 = 0 with respect to the parabola x^2 – 4x – 8y + 12 = 0.

Sol.

Given parabola is $x^2 - 4x - 8y + 12 = 0$.

Let (x_1, y_1) be the pole.

Equation of polar is $S_1 = 0$

$\Rightarrow xx_1 - 2(x + x_1) - 4(y + y_1) + 12 = 0$

$x(x_1 - 2) - 4y - 2x_1 - 4y_1 + 12 = 0$ …(i)

Comparing equation with equation

$2x - y - 4 = 0$

We get : $\frac{x_1 - 2}{2} = \frac{-4}{-1} = \frac{-2x_1 - 4y_1 + 12}{-4}$

$x_1 = 10$, $y_1 = 2$

Pole P = (10, 2).

3. Show that the lines 2x – y = 0 and 6x – 2y + 1 = 0 are conjugate lines with respect to the parabola y^2 = 2x.

Sol.

Equation of the parábola is $y^2 = 2x$

Pole of the line is $2x - y = 0$ is $\left(\frac{n}{\ell}, \frac{-2am}{\ell}\right) = \left(\frac{0}{2}, \frac{-2(1/2)(-1)}{2}\right)$

Pole = $\left(0, \frac{1}{2}\right)$

substitute $\left(0, \frac{1}{2}\right)$ in $6x - 2y + 1 = 0$

$\Rightarrow 6\cdot 0 - 2\cdot\frac{1}{2} + 1 = 0 \Rightarrow 0 = 0$

Hence $2x - y = 0$ and $6x - 2y + 1 = 0$ are conjugate lines.

4. **Find the value of k if $2x + 3y + 4 = 0$ and $x + y + k = 0$ are conjugate with respect to the parabola $y^2 = 8x$.**

Sol. Equation of the parabola is $y^2 = 8x \Rightarrow a = 2$

given $2x + 3y + 4 = 0$ and $x + y + k = 0$ are conjugate w.r.t. $y^2 = 8x$.

Therefore , $l_1n_2 + l_2n_1 = 2am_1m_2$

$l_1n_2 + l_2n_1 = 2k + 1(4)$

$2am_1m_2 = 2(2)(3)(1) = 12$

$2k + 4 = 12$

$2k = 8 \Rightarrow k = 4$

II.

1. **Find the equation of the chord of contact of the point A(2, 3) with respect to the parabola $y^2 = 4x$. Find the points where the chord of contact meets the parabola. Using these find the equations of tangents passing through A to the given parabola.**

Sol. Given parabola is $y^2 = 4x \Rightarrow a = 1$

Equation of chord of contact of A(2, 3) is $S_1 = 0$

$3y = 2(x + 2) \Rightarrow 2x - 3y + 4 = 0$ ----(1).

$\Rightarrow y = \dfrac{2x+4}{3}$ substitute this in $y^2 = 4x$

$$\Rightarrow \left(\frac{2x+4}{3}\right)^2 = 4x \Rightarrow x^2 - 5x + 4 = 0$$

$$\Rightarrow (x-4)(x-1) = 0 \Rightarrow x = 4, 1 \Rightarrow y = 4, 2$$

Equation of the tangent at (4, 4) is $S_1 = 0$

$y(4) = 2(x + 4) \Rightarrow x - 2y + 4 = 0$

Equation of the tangent at (1, 2) is $S_1 = 0$

$y(2) = 2(x + 1) \Rightarrow x - y + 1 = 0$.

2. **Prove that the polars of all points on the directrix of a parabola $y^2 = 4ax$ $(a > 0)$ are concurrent at focus**.

Sol. Equation of the parabola is $y^2 = 4ax$

Equation of the directrix is $x = -a$

Any point on the directrix is $P(-a, y_1)$

Polar of $P(-a, y_1)$ is $yy_1 = 2a(x - a)$

This polar always passes through the fixed point (a, 0) which is the focus of the parabola.

$\therefore$ The polars of all points on the directrix are concurrent at the focus of the parabola.

III.

1. **If the polar of P with respect to the parabola $y^2 = 4ax$, touches the circle $x^2 + y^2 = 4a^2$, then show that P lies on the curve $x^2 - y^2 = 4a^2$.**

Sol. Equation of the parabola is $y^2 = 4ax$

Let $P(x_1, y_1)$be the pole.

Polar of $P(x_1, y_1)$ is $S_1 = 0$

$yy_1 = 2a(x + x_1)$

$2ax - yy_1 + 2ax_1 = 0 \quad \ldots(1)$

If (1) is a tangent to this circle $x^2 + y^2 = 4a^2$ then

Length of the perpendicular form Centre is C(0, 0), = radius of the circle.

$$\frac{|0-0+2ax_1|}{\sqrt{4a^2+y_1^2}} = 2a$$

$$\Rightarrow \frac{|0-0+2ax_1|}{\sqrt{4a^2+y_1^2}} = 2a$$

Locus of $P(x_1, y_1)$ is $x^2 - y^2 = 4a^2$.

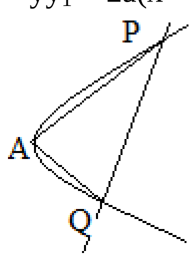
2. **Show that the poles of the chords of a parabola $y^2 = 4ax$ which subtend a right angle at vertex, lie on a line parallel to its directrix.**

Sol. Equation of the parabola is $y^2 = 4ax \qquad \ldots(1)$

Let $P(x_1, y_1)$ be the pole.

Polar of $P(x_1, y_1)$ is $S_1 = 0$

$yy_1 = 2a(x + x_1) = 2ax + 2ax_1$



$\Rightarrow yy_1 - 2ax = 2ax_1$

$$\Rightarrow \frac{yy_1 - 2ax}{2ax_1} = 1 \qquad \ldots(2)$$

Homogenising (1) with the help of (2)

$$y^2 = 4ax.1 = \frac{4ax(yy_1 - 2ax)}{2ax_1}$$

$\Rightarrow x_1y^2 = 2xyy_1 - 4ax^2 \Rightarrow 4ax^2 - 2y_1xy + x_1y^2 = 0$ but the lines are perpendicular, therefore

Co efficient of x^2 + Co efficient of y^2 =0

$4a + x_1 = 0 \quad \Rightarrow x_1 = -4a$

Locus of $P(x_1, y_1)$ is $x = -4a$, which is parallel to the directrix is $x = -a$.

3. Show that the chord of contact of any point on the line $x + 4a = 0$ with respect to parabola $y^2 = 4ax$ will subtends a right angle at the vertex.

Sol. Equation of the parabola is $y^2 = 4ax$ -----(1)

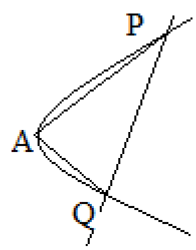
Any point on $x + 4a = 0$ is $P(-4a, y_1)$

Equation of the chord of contact of P is $S_1 = 0$

$yy_1 = 2ax - 8a^2$

$\Rightarrow 8a^2 = 2ax - yy_1$

$\Rightarrow \frac{2a - yy_1}{8a^2} = 1 \quad \ldots(2)$



Homogenising (1) with help of (2) combined equation of AQ, AR is

$$\Rightarrow y^2 = 4ax.1 = \frac{4ax(2ax - yy_1)}{8a^2} \Rightarrow 2ay^2 = 2ax^2 - xyy_1$$

$$\Rightarrow 2ax^2 - xyy_1 - 2ay^2 = 0$$

From above equation,

Coefficient of x^2 + coefficient of y^2

$= 2a - 2a = 0$

$\therefore \angle AQR = 90°$

$\Rightarrow$ QR subtends a right angle at the vertex.

4. Show that the poles of chords of the parabola $y^2 = 4ax$ which are at a constant distance 'a' from the focus lie on the curve $y^2 = 8ax + 4x^2$.

Sol. Equation of parabola is $y^2 = 4ax$

Focus $S = (a, 0)$

LET $P(x_1, y_1)$ be the pole.

Polar of $P(x_1, y_1)$ is $S_1 = 0$

$yy_1 = 2a(x + x_1) = 2ax + 2ax_1$

$\Rightarrow 2ax - yy_1 + 2ax_1 = 0$

Given that the perpendicular distance from S to this line = a

$$\Rightarrow a = \frac{|2a^2 - 0 + 2ax_1|}{\sqrt{4a^2 + y_1^2}} = \frac{2a|a + x_1|}{\sqrt{4a^2 + y_1^2}}$$

$$\Rightarrow 4a^2 + y_1^2 = 4(a + x_1)^2$$

$$\Rightarrow 4a^2 + y_1^2 = 4a^2 + 4x_1^2 + 8ax_1$$

Locus of $P(x_1, y_1)$ is $y^2 = 8ax + 4x^2$.

PROBLEMS FOR PRACTICE

1. **Find the coordinates of the vertex and focus, and the equations of the directrix and axes of the following parabolas.**
   i) $y^2 = 16x$ ii) $x^2 = -4y$
   iii) $3x^2 - 9x + 5y - 2 = 0$
   iv) $y^2 - x + 4y + 5 = 0$
2. **Find the equation of the parabola whose vertex is (3, –2) and focus is (3, 1).**

Ans. $(x - 3)^2 = 12(y + 2)$

3. **Find the coordinates of the points on the parabola $y^2 = 2x$ whose focal distance is 5/2.**

Ans. (2, 2) and (2, –2)

4. **Find the equation of the parabola passing through the points (–1, 2), (1, –1) and (2, 1) and having its axis parallel to the x-axis.**

Ans. $7y^2 - 3y + 6x - 16 = 0$

5. **A double ordinate of the curve $y^2 = 4ax$ is of length 8a. Prove that the line from the vertex to its ends are at right angles.**

Sol. Let $P = (at^2, 2at)$ and $P' = (at^2, -2at)$ be the ends of double ordinate PP′. Then

$$8a = PP' = \sqrt{0 + (4at)^2} = 4at \Rightarrow t = 2$$

$\therefore$ P = (4a, 4a), P′ = (4a, –4a)

Slope of $\overline{AP} \times$ slope of $\overline{AP'}$

$$= \left(\frac{4a}{4a}\right)\left(-\frac{4a}{4a}\right) = -1$$

$$\therefore \angle PAP' = \frac{\pi}{2}$$

6. **(i) If the coordinates of the ends of a focal chord of the parabola $y^2 = 4ax$ are (x_1, y_1) and (x_2, y_2), then prove that $x_1x_2 = a^2$, $y_1y_2 = -4a^2$.**

(ii) For a focal chord PQ of the parabola $y^2 = 4ax$, if SO = l and SQ = l' then prove that $\frac{1}{l}+\frac{1}{l'}=\frac{1}{a}$.

Sol. i) Let $P(x_1,y_1) = (at_1^2, 2at_1)$ and $Q(x_2, y_2) = (at_2^2, 2at_2)$ be two end points of a focal chord.

P, S, Q are collinear.

Slope of $\overline{PS}$ = Slope of $\overline{QS}$

$$\frac{2at_1}{at_1^2-a}=\frac{2at_2}{at_2^2-a}$$

$$t_1t_2^2-t_1=t_2t_1^2-t_2$$

$$t_1t_2(t_2-t_1)+(t_2-t_1)=0$$

$$1+t_1t_2=0\Rightarrow t_1t_2=-1$$

From (1)

$$x_1x_2=at_1^2at_2^2=a^2(t_2t_1)^2=a^2$$

$$y_1y_2=2at_12at_2=4a^2(t_2t_1)=-4a^2$$

ii) Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the extremities of a focal chord of the parabola, then $t_1t_2 = -1$ (from(1))

$$l=SP=\sqrt{(at_1^2-a)^2+(2at_1-0)^2}$$

$$=a\sqrt{(t_1^2-1)^2+4t_1^2}=a(1+t_1^2)$$

$$l'=SQ=\sqrt{(at_2^2-a)^2+(2at_2-0)^2}$$

$$=a\sqrt{(t_2^2-1)^2+4t_2^2}=a(1+t_2^2)$$

$$\therefore\ (l-a)(l'-a)=a^2t_1^2t_2^2=a^2(t_1t_2)^2=a^2$$

$$[\because t_1t_2=-1]$$

$$ll'-a(l+l')=0\Rightarrow\frac{1}{l}+\frac{1}{l'}=\frac{1}{a}$$

7. **If Q is the foot of the perpendicular from a point P on the parabola $y^2 = 8(x - 3)$ to its directrix. S is the focus of the parabola and if SPQ is an equilateral triangle then find the length of side of the triangle.**

Ans. 8

8. Find the condition for the straight line
$lx + my + n = 0$ to be a tangent to the parabola $y^2 = 4ax$ and find the coordinates of the point of contact.

Ans. $\left(\frac{n}{l}, \frac{-2am}{l}\right)$

9. Show that the straight line $7x + 6y = 13$ is a tangent to the parabola $y^2 - 7x - 8y + 14 = 0$ and find the point of contact.

Ans. (1, 1)

10. Prove that the normal chord at the point other than origin whose ordinate is equal to its abscissa subtends a right angle at the focus.

Sol. Let the equation of the parabola be $y^2 = 4ax$ and $P(at^2, 2at)$ be any point …(1)

On the parabola for which the abscissa is equal to the ordinate.

i.e. $at^2 = 2at \Rightarrow t = 0$ or $t = 2$. But $t \neq 0$.

Hence the point (4a, 4a) at which the normal is

$y + 2x = 2a(2) + a(2)^3$

$y = (12a - 2x)$ …(2)

Substituting the value of

$y = (12a - 2x)$ in (1) we get

$(12a - 2x)^2 = 4ax$

$x^2 - 13ax + 36a^2 = (x - 4a)(x - 9a) = 0$

$\Rightarrow x = 4a, 9a$

Corresponding values of y are 4a and –6a.

Hence the other points of intersection of that normal at P(4a, 4a) to the given parabola is Q(9a, –6a), we have S(a, 0).

Slope of the $\overline{SP} = m_1 = \frac{4a - 0}{4a - a} = \frac{4}{3}$

Slope of the $\overline{SQ} = m_2 = \frac{-6a - 0}{9a - a} = -\frac{3}{4}$

Clearly $m_1 m_2 = -1$, so that $\overline{SP} \perp \overline{SQ}$.

11. From an external point P, tangent are drawn to the parabola $y^2 = 4ax$ and these tangent make angles θ_1, θ_2 with its axis, such that $\tan\theta_1 + \tan\theta_2$ is constant b. Then show that P lies on the line $y = bx$.

12. Show that the common tangent to the parabola y^2 = 4ax and x^2 = 4by is

$xa^{1/3} + yb^{1/3} + a^{2/3}b^{2/3} = 0$.

Sol. The equations of the parabolas are

$y^2 = 4ax$ …(1) and

$x^2 = 4by$ …(2)

Equation of any tangent to (1) is of the form

$y = mx + \frac{a}{m}$ …(3)

If the line (3) is a tangent to (2) also, we must get only one point of intersection of (2)and (3).

Substituting the value of y from (3) in (2), we get $x^2 = 4b\left(mx + \frac{a}{m}\right)$ is $mx^2 - 4bm^2x - 4ab = 0$ should have equal roots therefore its discrimi-nent must be zero. Hence

$16b^2m^4 - 4m(-4ab) = 0$

$16b\,(bm^4 + am) = 0$

$m(bm^3 + a) = 0$, but $m \neq 0$

$\therefore$ $m = -a^{1/3}b^{1/3}$ substituting in (3) the equation of the common tangent becomes

$y = -\left(\frac{a}{b}\right)^{1/3} x + \frac{a}{\left(-\frac{a}{b}\right)^{1/3}}$ or

$a^{1/3}x + b^{1/3}y + a^{2/3}b^{2/3} = 0$.

13. Prove that the area of the triangle formed by the tangents at (x_1, y_1), (x_2, y_2) and (x_3, y_3) to the parabola y^2 = 4ax (a > 0) is $\frac{1}{16a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$ **sq.units.**

Sol. Let $D(x_1, y_1) = (at_1^2, 2at_1)$

$E(x_2, y_2) = (at_2^2, 2at_2)$ and

$F(x_3, y_3) = (at_3^2, 2at_3)$

Be three point on the parabola.

$y^2 = 4ax$ (a > 0)

The equation of the tangents at D, E and F are

$t_1y = x + at_1^2$ ...(1)

$t_2y = x + at_2^2$ ...(2)

$t_3y = x + at_3^2$ ...(3)

(1) – (2) $\Rightarrow$ $(t_1 - t_2)y = a(t_1 - t_2)(t_1 + t_2)$

$\Rightarrow$ $y = a(t_1 + t_2)$ substituting in (1) we get,

$x = at_1t_2$

$\therefore$ The point of intersection of the tangents at D and E is say $P[at_1t_2, a(t_1+t_2)]$

Similarly the points of intersection of tangent at E, F and at F, D are $Q[at_2t_3, a(t_2+t_3)]$ and $R[at_3t_1, a(t_3+t_1)]$ respectively.

Area of ΔPQR

$$= \text{Absolute value of } \frac{1}{2}\begin{vmatrix} at_1t_2 & a(t_2+t_2) & 1 \\ at_2t_3 & a(t_2+t_3) & 1 \\ at_1t_3 & a(t_1+t_3) & 1 \end{vmatrix}$$

$$= \text{Absolute value of } \frac{a^2}{2}\begin{vmatrix} t_1t_2 & t_2+t_2 & 1 \\ t_2t_3 & t_2+t_3 & 1 \\ t_1t_3 & t_1+t_3 & 1 \end{vmatrix}$$

$$= \text{Absolute value of } \frac{a^2}{2}\begin{vmatrix} t_1(t_2-t_3) & t_2-t_3 & 0 \\ t_3(t_2-t_1) & t_2-t_1 & 0 \\ t_1t_3 & t_1+t_3 & 1 \end{vmatrix}$$

= Absolute value of

$$\frac{a^2}{2}(t_2-t_3)(t_2-t_1)\begin{vmatrix} t_1 & 1 & 0 \\ t_3 & 1 & 0 \\ t_1t_3 & t_1+t_3 & 1 \end{vmatrix}$$

$$= \frac{a^2}{2}|(t_2-t_3)(t_2-t_1)(t_1-t_3)|$$

$$= \frac{1}{16a}|2a(t_1-t_2)2a(t_2-t_3)2a(t_3-t_1)|$$

$$= \frac{1}{16a}|(y_1-y_2)(y_2-y_3)(y_3-y_1)| \text{ sq. units.}$$

14. Find the value of k if

i) Points (1, 2) (k – 1) are conjugate with respect to the parabola $y^2 = 8x$.

ii) The line $x + y + 2 = 0$ and $x - 2y + k = 0$ are conjugate with respect to the parabola $y^2 + 4x - 2y - 3 = 0$.

Ans. (i) –3/2, (ii) 1

15. Prove that the poles of normal chord of the parabola $y^2 = 4ax$ lie on the curve$(x + 2a)y^2 + 4a^3 = 0$.

16. Prove that the poles of tangents to the parabola $y^2 = 4ax$ with respect to the parabola $y^2 = 4bx$ lie on a parabola.

Sol. Equation of any tangent to $y^2 = 4ax$ is of the form $y = mx + \frac{a}{m}$ …(1)

Let $P(x_1, y_1)$ be the pole of (1) w.r.t. $y^2 = 4bx$

Then the polar of $P(x_1, y_1)$ w.r.t $y^2 = 4bx$ is :

$yy_1 = 2b(x + x_1)$

$\therefore$ (1) and (2) represent the same line

Comparing the coefficients

$$\frac{y_1}{1} = \frac{2b}{m} = \frac{2bx_1m}{a} \Rightarrow m^2 = \frac{a}{x_1} m = \frac{2b}{y_1}$$

Eliminating m,

$$\frac{4b^2}{y_1^2} = \frac{a}{x_1} \Rightarrow y_1^2 = \frac{4b^2}{a} x_1$$

$\therefore$ The pole $P(x_1, y_1)$ lies on the parabola is :

$$y^2 = \frac{4b^2}{a} x$$

17. If the normal at t_1 and t_2 to the parabola $y^2 = 4ax$ meet on the parabola, then show that $t_1t_2 = 2$.

Proof :

Let the normals at t_1 and t_2 meet at t_3 on the parabola.

The equation of the normal at t_1 is :

$y + xt_1 = 2at_1 + at_1^3$ …(1)

Equation of the chord joining t_1 and t_3 is :

$y(t_1 + t_3) = 2x + 2at_1t_3$ …(2)

(1) and (2) represent the same line

$$\therefore \frac{t_1 + t_3}{1} = \frac{-2}{t_1} \Rightarrow t_3 = -t_1 - \frac{2}{t_1}$$

Similarly $t_3 = -t_2 - \frac{2}{t_2}$

$$\therefore -t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2} \Rightarrow t_1 - t_2 = \frac{2}{t_2} - \frac{2}{t_1}$$

$$\Rightarrow t_1 - t_2 = \frac{2(t_1 - t_2)}{t_1t_2} \Rightarrow t_1t_2 = 2.$$