CHAPTER 3

PARABOLA

TOPICS:

1.Conic Sections

2. Standard Form Of A Parabola, Nature Of The Curve And Properties

3.Tangents And Normals

4.Chord And Chord Of Contact

5.Parametric Equation

6.Pole And Polar

7.Conjugate Points And Conjugate Lines.

PARABOLA

CONIC

The locus of a point which moves in a plane so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line is called a conic section or conic. The fixed point is called focus, the fixed straight line is called directrix and the constant ratio ‘e’ is called eccentricity of the conic.

i) If $e = 1$, then the conic is called a parabola.
ii) If $e < 1$, then the conic is called an ellipse.
iii) If $e > 1$, then the conic is called a hyperbola.

Note.

The equation of a conic is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

DIRECTRIX OF THE CONIC

A line $L = 0$ passing through the focus of a conic is said to be the principal axis of the conic if it is perpendicular to the directrix of the conic.

VERTICES

The points of intersection of a conic and its principal axis are called vertices of the conic.

CENTRE

The midpoint o the line segment joining the vertices of a conic is called centre of the conic.

Note 1 : If a conic has only one vertex then its centre coincides with the vertex.

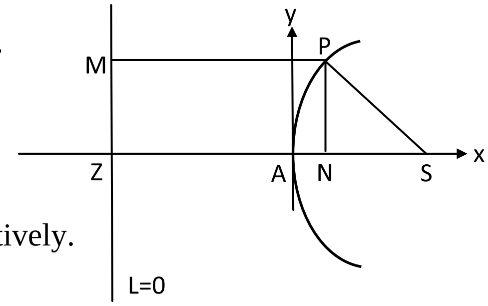
Note 2 : If a conic has two vertices then its centre does not coincide either of the vertices. In this case the conic is called a central conic.

STANDARD FORM

A conic is said to be in the standard form if the principal axis of the conic is x-axis and the centre of the conic is the origin.

EQUATION OF A PARABOLA IN STANDARD FORM.

The equation of a parabola in the standard form is $y^2 = 4ax$.



Proof

Let S be the focus and L = 0 be the directrix of the parabola.

Let P be a point on the parabola.

Let M, Z be the projections of P, S on the directrix L = 0 respectively.

Let N be the projection of P on SZ.

Let A be the midpoint of SZ.

Therefore, SA = AZ, $\Rightarrow$ A lies on the parabola. Let AS = a.

Let AS, the principal axis of the parabola as x-axis and Ay perpendicular to SZ as y-axis.

Then S = (a, 0) and the parabola is in the standard form.

Let P = (x_1, y_1).

Now PM = NZ = NA + AZ = $x_1 + a$

P lies on the parabola $\Rightarrow \dfrac{PS}{PM} = 1 \Rightarrow PS = PM$

$$\Rightarrow \sqrt{(x_1 - a)^2 + (y_1 - 0)^2} = x_1 + a$$

$$\Rightarrow (x_1 - a)^2 + y_1^2 = (x_1 + a)^2$$

$$\Rightarrow y_1^2 = (x_1 + a)^2 - (x_1 - a)^2 \Rightarrow y_1^2 = 4ax_1$$

The locus of P is $y^2 = 4ax$.

$\therefore$ The equation to the parabola is $y^2 = 4ax$.

NATURE OF THE CURVE $y^2 = 4ax$.

i) The curve is symmetric with respect to the x-axis.

$\therefore$ The principal axis (x-axis) is an axis of the parabola.

ii) $y = 0 \Rightarrow x = 0$. Thus the curve meets x-axis at only one point (0, 0).

Hence the parabola has only one vertex.

iii) If $x<0$ then there exists no $y \in R$. Thus the parabola does not lie in the second and third quadrants.

iv) If $x > 0$ then $y^2 > 0$ and hence y has two real values (positive and negative). Thus the parabola lies in the first and fourth quadrants.

v) $x = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0, 0$. Thus y-axis meets the parabola in two coincident points and hence y-axis touches the parabola at (0, 0).

vi) as $x \to \infty \Rightarrow y^2 \to \infty \Rightarrow y \to \pm\infty$

Thus the curve is not bounded (closed) on the right side of the y-axis.

DOUBLE ORDINATE

A chord passing through a point P on the parabola and perpendicular to the principal axis of the parabola is called the double ordinate of the point P.

FOCAL CHORD

A chord of the parabola passing through the focus is called a focal chord.

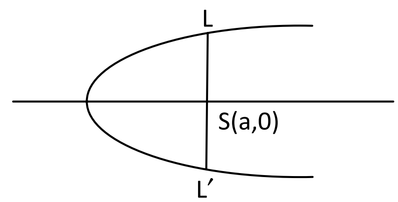
LATUS RECTUM

A focal chord of a parabola perpendicular to the principal axis of the parabola is called latus rectum. If the latus rectum meets the parabola in L and L′, then LL′ is called length of the latus rectum.

THEOREM

The length of the latus rectum of the parabola $y^2 = 4ax$ is 4a.

Proof :



Let LL′ be the length of the latus rectum of the parabola $y^2 = 4ax$.

Let SL = l, then L = (a, l).

Since L is a point on the parabola $y^2 = 4ax$, therefore $l^2 = 4a(a)$

$\Rightarrow l^2 = 4a^2 \Rightarrow l = 2a \Rightarrow SL = 2a$

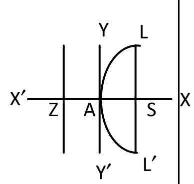
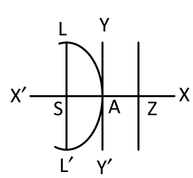
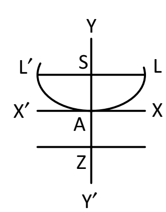
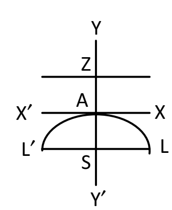
$\therefore LL' = 2SL = 4a$.

FOCAL DISTANCE

If P is a point on the parabola with focus S, then SP is called focal distance of P.

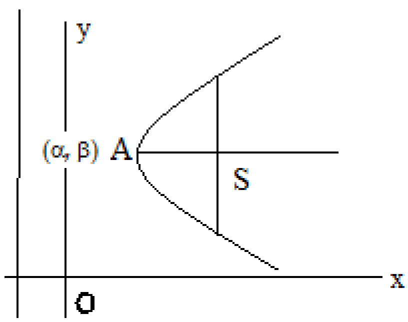
THEOREM

The focal distance of $P(x_1, y_1)$ on the parabola $y^2 = 4ax$ is $\mathbf{x_1 + a}$.

SL.NO	CONTENT	I	II	III	IV
	EQUATION	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
	Figure				
1.	VERTEX	(0, 0)	(0, 0)	(0, 0)	(0, 0)
2.	FOCUS	(a, 0)	(–a, 0)	(0, a)	(0, –a)
3.	ENDS OF LATUSRECTUM	(a, ± 2a)	(–a, ± 2a)	(± 2a, a)	(± 2a, –a)

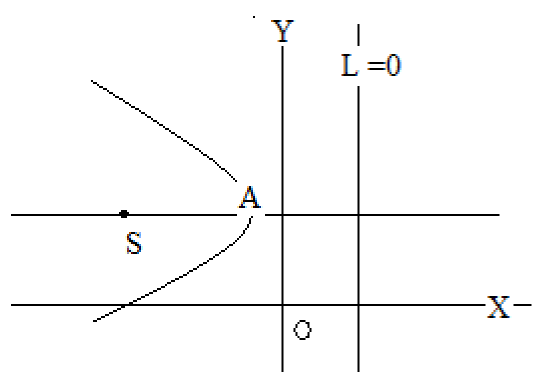
4.	EQUATION OF AXIS	Y=0	Y=0	X=0	X=0
5.	EQUATION OF DIRETRIX	x = –a	x = a	y = –a	y = a
6.	TANGENT AT VERTEX	X =0	X= 0	Y=O	Y=0
7.	EQUATION OF LATUSRECTUM	X=a	X=-a	Y=a	Y=-a
8.	LENGTH OF LATUSRECTUM	4a	4a	4a	4a
9.	DISTANCE FROM FOCUS TO DIRECTRIX	2a	2a	2a	2a

1. For the parabola $(y - \beta)^2 = 4a(x - \alpha)$



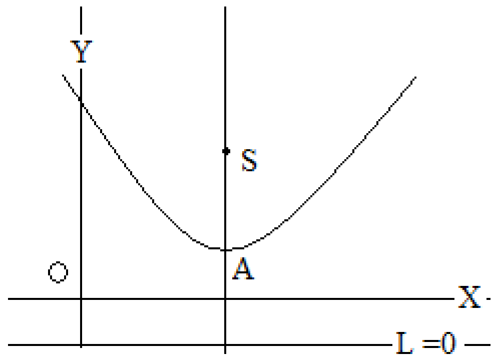
1) vertex A= (α, β) 2) FocusS $(a+\alpha, \beta)$ 3) directrix is L=x - α = -a 4) latusrectum = |4a| 5. axis of the parabola $y = \beta$

2. For the parabola $(y - \beta)^2 = -4a(x - \alpha)$



1) vertex A= (α, β) 2) FocusS $(\alpha\text{-}a, \beta)$ 3) directrix isL= x - α = a 4) latusrectum = |4a| 5) axis of the parabola $y = \beta$.

3. For the parabola $(x - \alpha)^2 = 4a(y - \beta)$

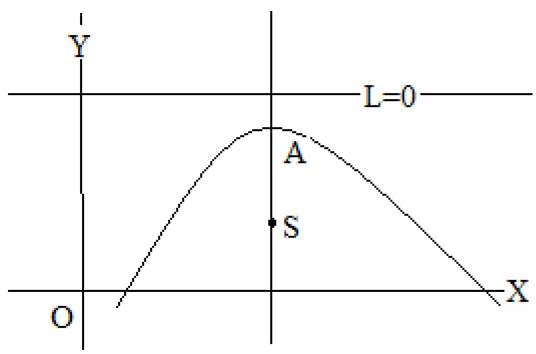


1) vertex $A = (\alpha, \beta)$ 2) FocusS $(\alpha, a+\beta)$

3) directrix is $L = y - \beta = -a$ 4) latusrectum = $|4a|$

5) axis of the parabola is $x = \alpha$

4. For the parabola $(x - \alpha)^2 = - 4a(y - \beta)$



1) vertex $A = (\alpha, \beta)$ 2) FocusS $(\alpha, \beta - a)$

3) directrix is $L = y - \beta = a$ 4) latusrectum = $|4a|$

5) axis of the parabola is $x = \alpha$

Notation : We use the following notation in this chapter

$S \equiv y^2 - 4ax$

$S_1 \equiv yy_1 - 2a(x + x_1)$

$S_{11} = S(x_1, y_1) \equiv y_1^2 - 4ax_1$

$S_{12} \equiv y_1y_2 - 2a(x_1 + x_2)$

Note :

Let $P(x_1, y_1)$ be a point and $S \equiv y^2 - 4ax = 0$ be a parabola. Then

i) P lies on the parabola $\Leftrightarrow S_{11} = 0$

ii) P lies inside the parabola $\Leftrightarrow S_{11} < 0$

iii) P lies outside the parabola $\Leftrightarrow S_{11} > 0$.

EXERCISE 3(a)

1. Find the vertex and focus of $4y^2 + 12x - 20y + 67 = 0$.

Sol. Given parábola $4y^2 + 12x - 20y + 67 = 0$

$4y^2 - 20y = -12x - 67$

$$y^2 - 5y = -3x - \frac{67}{4}$$

$$\Rightarrow \left(y - \frac{5}{2}\right)^2 - \frac{25}{4} = -3x - \frac{67}{4}$$

$$\Rightarrow \left(y - \frac{5}{2}\right)^2 = -3x - \frac{42}{4} = -3\left(x + \frac{7}{2}\right)$$

$$\Rightarrow \left(y - \frac{5}{2}\right)^2 = -3\left[x - \left(-\frac{7}{2}\right)\right]$$

$$\therefore h = -\frac{7}{2},\ k = \frac{5}{2},\ a = -\frac{3}{4}$$

Vertex A is $\left(-\frac{7}{2}, \frac{5}{2}\right)$

Focus is s (h+a, k) = $\left(-\frac{7}{2} - \frac{3}{4}, \frac{5}{2}\right) = \left(\frac{-17}{4}, \frac{5}{2}\right)$

2. Find the vertex and focus of $x^2 - 6x - 6y + 6 = 0$.

Sol. Given parabola is $x^2 - 6x - 6y + 6 = 0$

$\Rightarrow x^2 - 6x = 6y - 6$

$\Rightarrow (x - 3)^2 - 9 = 6y - 6$

$\Rightarrow (x - 3)^2 = 6y + 3$

$$\Rightarrow (x - 3)^2 = 6\left(y + \frac{1}{2}\right) = 6\left[y - \left(\frac{-1}{2}\right)\right]$$

$\therefore$ h = 3, k = $\frac{-1}{2}$, a = $\frac{6}{4}=\frac{3}{2}$

Vertex = (h, k) = $\left(3,\frac{-1}{2}\right)$

Focus = (h, k+a) = $\left(3,-\frac{1}{2}-\frac{1}{2}\right)=(3,-1)$

3. Find the equations of axis and directrix of the parabola $y^2 + 6y - 2x + 5 = 0$.

Sol. Given parabola is $y^2 + 6y = 2x - 5$

$\Rightarrow [y-(-3)]^2 - 9 = 2x - 5$

$\Rightarrow [y-(-3)]^2 = 2x - 5 + 9$

$\Rightarrow [y-(-3)]^2 = 2x + 4$

$\Rightarrow [y-(-3)]^2 = 2[x-(-2)]$

Comparing with $(y-k)^2 = 4a(x-h)$ we get,

(h, k) = (–2, –3), a = $\frac{1}{2}$

Equation of the axis y – k = 0 i.e. y + 3 = 0

Equation of the directrix x – h + a = 0

i.e. $x-(-2)+\frac{1}{2}=0$

2x + 5 = 0.

4. Find the equation of axis and directrix of the parabola $4x^2 + 12x - 20y + 67 = 0$.

Sol. Given parabola $4x^2 + 12x - 20y + 67 = 0$

$\Rightarrow 4x^2 + 12x = 20y - 67$

$\Rightarrow x^2 + 3x = 5y - \frac{67}{4}$

$\Rightarrow \left(x+\frac{3}{2}\right)^2-\frac{9}{4}=5y-\frac{67}{4}$

$\Rightarrow \left(x+\frac{3}{2}\right)^2=5y-\frac{58}{4}=5y-\frac{29}{2}$

$\Rightarrow \left[x-\left(-\frac{3}{2}\right)\right]^2=5\left[y-\frac{29}{10}\right]$

Comparing with $(x-h)^2 = 4a(y-k)$

$(h, k) = \left(-\frac{3}{2}, \frac{29}{10}\right)$, $a = \frac{5}{4}$

Equation of the axis x – h = 0

i.e. $x + \frac{3}{2} = 0 \Rightarrow 2x + 3 = 0$

Equation of the directrix, y – k + a = 0

$$y - \frac{29}{10} + \frac{5}{4} = 0 \Rightarrow 20y - 33 = 0$$

5. Find the equation of the parabola whose focus is s(1, –7) and vertex is A(1, –2).

Sol.

Focus s = (1, -7), vertex A(1, –2)

h = 1, k = –2, a = –2 + 7 = 5

since x coordinates of S and A are equal, axis of the parabola is parallel to y-axis.

And the y coordinate of S is less than that of A, therefore the parabola is a down ward parabola.

Let equation of the parabola be

$(x - h)^2 = -4a(y - k)$

$\Rightarrow (x - 1)^2 = -20(y + 2)$

$\Rightarrow x^2 - 2x + 1 = -20y - 40$

$\Rightarrow x^2 - 2x + 20y + 41 = 0$

6. Find the equation of the parabola whose focus is S(3, 5) and vertex is A(1, 3).

Sol.

Focus S(3, 5) and vertex A(1, 3)`

let Z (x, y) be the projection of S on directrix. The A is the mid point of SZ.

$$\Rightarrow (1, 3) = \left(\frac{3+x}{2}, \frac{5+y}{2}\right) \Rightarrow x = -1, y = 1$$

Z =(-1, 1)

Slope of directrix = -1/(slope of SA) $= \frac{-1}{\left(\frac{5-3}{3-1}\right)} = -1$

Equation of directrix is y-1 = -1 (x+1) i.e., x + y =0 ----(1)

let P (x,y) be any point on the parabola. Then

$SP = PM \Rightarrow SP^2 = PM^2$ where PM is the perpendicular from P the directrix.

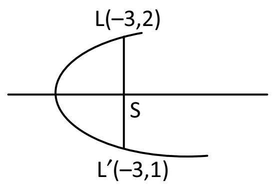
$$\Rightarrow (x - 3)^2 + (y - 5)^2 = \frac{(x + y)^2}{1+1}$$

$\Rightarrow 2(x^2 - 6x + 9 + y^2 - 10y + 25) = (x + y)^2$

$\Rightarrow 2x^2 + 2y^2 - 12x - 20y + 68 = x^2 + 2xy + y^2$

i.e. $x^2 - 2xy + y^2 - 12x - 20y + 68 = 0$.

7. Find the equation of the parabola whose latus rectum is the line segment of joining the points (–3, 2) and (–3, 1).



Sol. Ends of the latus rectum are $L(-3, 2)$ and $L'(-3, 1)$.

Length of the latusrectum is $LL' = \sqrt{(-3+3)^2 + (2-1)^2} = \sqrt{0+1} = 1$ (= 4a)

$\Rightarrow 4|a| = 1 \Rightarrow |a| = \frac{1}{4} \Rightarrow a = \pm\frac{1}{4}$

S is the midpoint of LL′ $\Rightarrow S = \left(-3, \frac{3}{2}\right)$

Case I : a = –1/4

$\Rightarrow A = \left[-3 + \frac{1}{4}, \frac{3}{2}\right]$

Equation of the parabola is $\left(y - \frac{3}{2}\right)^2 = -\left(x + 3 - \frac{1}{4}\right)$

$\Rightarrow \frac{(2y-3)^2}{4} = \frac{-(4x+12-1)}{4}$

$\Rightarrow (2y-3)^2 = -(4x+11)$

Case II : a = 1/4

$\Rightarrow A = \left[-3 - \frac{1}{4}, \frac{3}{2}\right]$

Equation of the parabola is $\left(y - \frac{3}{2}\right)^2 = \left(x + 3 + \frac{1}{4}\right)$

$\Rightarrow \frac{(2y-3)^2}{4} = \frac{(4x+12-1)}{4} \Rightarrow (2y-3)^2 = 4x + 13$

8. Find the position (interior or exterior or on) of the following points with respect to the parabola $y^2 = 6x$. (i) (6, –6), (ii) (0, 1), (iii) (2, 3)

Sol. Equation of the parabola is $y^2 = 6x$

$\Rightarrow S \equiv y^2 - 6x$

i) $S_{11} = (-6)^2 - 6.6 = 36 - 36 = 0$

$\therefore$ (6, –6) lies on the parabola.

ii) (0, 1)

$S_{11} = 1^2 - 6.0 = 1 > 0$

$\therefore$ (0, 1) lies outside the parabola.

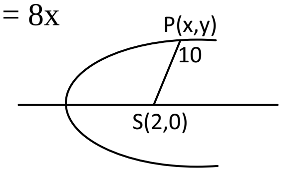
iii)(2, 3)

$S_{11} = 3^2 - 6.2 = 9 - 12 = -3 < 0$

$\therefore$ (2, 3) lies inside the parabola.

9. Find the coordinates of the point on the parabola $y^2 = 8x$ whose focal distance is 10.

Sol. Equation of the parabola is $y^2 = 8x$



$4a = 8 \Rightarrow a = 2$

$\Rightarrow S = (2, 0)$

let P(x, y) be a point on the parabola

Given SP = 10

$\Rightarrow |x + a| = 10 \Rightarrow x + 2 = \pm 10$

$\Rightarrow x = 8$ or -12

Case I : x = 8

$y^2 = 8x = 8 \times 8 = 64$

$y = \pm 8$

Coordinates of the required points are (8, 8) and (8, –8)

Case II : x = –12

$y^2 = 8 \times -12 = -96 < 0$

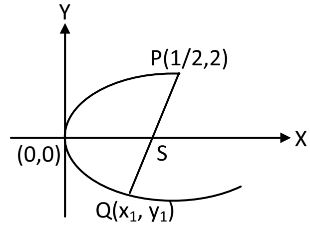
y is not real.

10. If (1/2, 2) is one extremity of a focal chord of the parabola $y^2 = 8x$. Find the coordinates of the other extremity.

Sol. Given parabola $y^2 = 8x$

focus $S = (2, 0)$

One end of the focal chord is $P\left(\frac{1}{2}, 2\right)$,



Let $Q = (x_1, y_1)$ be the other end of the focal chord.

Q is a point on the parabola, $y_1^2 = 8x_1 \Rightarrow x_1 = \frac{y_1^2}{8}$

$\Rightarrow Q = \left(\frac{y_1^2}{8}, y_1\right)$

Slope of SP = $\frac{0-2}{2-\frac{1}{2}} = \frac{-4}{3}$

Slope of SQ = $\frac{y_1 - 0}{\frac{y_1^2}{8} - 2} = \frac{8y_1}{y_1^2 - 16} = \frac{-4}{3}$

PSQ is a focal chord $\Rightarrow$ the points P, S, Q are collinear.

Therefore , Slope of SP = Slope of SQ

$24y_1 = -4y_1^2 + 64 \Rightarrow 4y_1^2 + 24y_1 - 64 = 0$

$\Rightarrow y_1^2 + 6y_1 - 16 = 0 \Rightarrow (y_1 + 8)(y_1 - 2) = 0$

$y_1 = 2, -8$; $x_1 = \frac{1}{2}, 8$

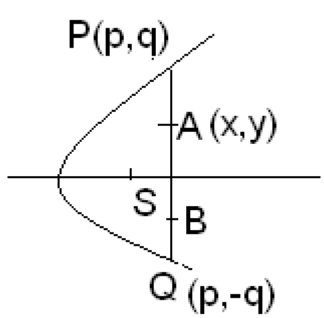
Therefore (8, –8) other extremity.

(If $x_1 = \frac{1}{2}$ then $y_1 = 2$ which is the given point .)

II.

1. Find the locus of the points of trisection of double ordinate of a parabola $y^2 = 4ax$ (a > 0).

Sol.



Given parabola is $y^2 = 4ax$

let P(p,q) and Q(p, –q) be the ends of the double ordinate.

Let A,B be the points of trisection of the double ordinate.

A divides PQ in the ratio 1 : 2.

$\Rightarrow A = \left(p, \frac{-q+2q}{3}\right) = \left(p, \frac{q}{3}\right)$

Let (x_1, y_1) be the coordinates of the one of the points of trisection , say A

Then $p = x_1$ and

$y_1 = \frac{q}{3} \Rightarrow q = 3y_1$

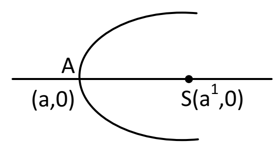
But $P(p,q) = (x_1, 3y_1)$ is a point on the parabola.

$\Rightarrow 4ax_1 = 9y_1^2$

Locus of (x_1,y_1) is $9y^2 = 4ax$.

2. Find the equation of the parabola whose vertex and focus are on the positive x-axis at a distance of a and a′ from origin respectively.

Sol. Vertex A (a, 0) and focus S (a′, 0)



AS = a′ – a

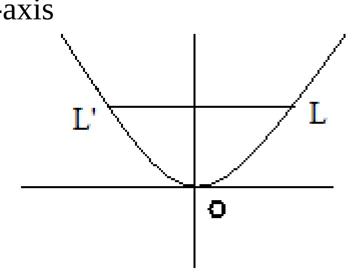
latusrectum = 4 (a′ – a)

Equation of the parabola is $y^2 = 4(a' - a)(x - a)$

3. If L and L′ are the ends of the latus rectum of the parabola $x^2 = 6y$. Find the equations of OL and OL′ where O is the origin. Also find the angle between them.

Sol. GIVEN parabola is $x^2 = 6y$

Curve is symmetric about Y-axis



$4a = 6 \Rightarrow a = \frac{3}{2}$

$L = (2a,\ a) = \left(3, \frac{3}{2}\right)$ and $L' = (-2a,\ a) = \left(-3, \frac{3}{2}\right)$

Now equation of OL is $x = 2y$

And equation of OL′ is $x = -2y$

Let θ be the angle between the lines, then

$$\tan\theta = \left|\frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{4}}\right| = \frac{4}{3} \Rightarrow \theta = \text{Tan}^{-1}\left(\frac{4}{3}\right)$$

4. **Find the equation of the parabola whose axis is parallel to x-axis and which passes through these points. A(–2, 1), B(1, 2), C(–1, 3)**

Sol.

Given that axis of the parabola is parallel to X-axis,

Let the equation of the parabola be $x = ay^2 + by + c$

It is Passing through (–2, 1), (1, 2) , (–1, 3)

(–2, 1) $\Rightarrow -2 = a + b + c$ …(i)

(1, 2) $\Rightarrow 1 = 4a + 2b + c$ …(ii)

(–1, 3) $\Rightarrow -1 = 9a + 3b + c$ …(iii)

(ii) – (iii) $2 = -5a - b$

(ii) – (i) $3 = 3a + b$

$5 = -2a$

$$a = -\frac{5}{2}, b = \frac{21}{2}, c = -10$$

$$x = -\frac{5}{2}y^2 + \frac{21}{2}y - 10$$

$$5y^2 + 2x - 21y + 20 = 0$$

5. **Find the equation of the parabola whose axis is parallel to Y-axis and which passes through the points (4, 5), (–2, 11), (–4, 21).**

Sol.

Given that axis of the parabola is parallel to X-axis,

Let the equation of the parabola be $y = ax^2 + bx + c$

It is Passing through (4, 5), (–2, 11), (–4, 21)

(4, 5) $\Rightarrow 5 = 16a + 4b + c$ …(i)

(–2, 11) $\Rightarrow 11 = 4a - 2b + c$ …(ii)

(–4, 21) $\Rightarrow 21 = 16a - 4b + c$ …(iii)

(ii) – (i) we get : $6 = -12 - 6b$

(iii) – (ii) : $10 = 12a - 2b$

Solving these equations ,

$b = -2$, $a = 1/2$, $c = 5$

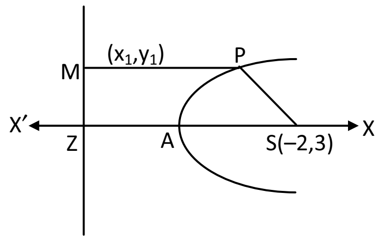
$$y = \frac{1}{2}x^2 - 2x + 5$$

$x^2 - 2y - 4x + 10 = 0$

III.

1. Find the equation of the parabola whose focus is (–2, 3) and directrix is the line 2x + 3y – 4 = 0. Also find the length of the latus rectum and the equation of the axis of the parabola.

Sol.



Focus S(–2, 3)

Equation of the directrix is 2x + 3y – 4 = 0.

Let $P(x_1, y_1)$ be any point on the parabola.

$SP^2 = (x_1 + 2)^2 + (y_1 - 3)^2$

Let PM be the perpendicular from P to the directrix.

From Def. of parabola $SP = PM \Rightarrow SP^2 = PM^2$

$$(x_1 + 2)^2 + (y_1 - 3)^2 = \frac{(2x_1 + 3y_1 - 4)^2}{13}$$

$$13(x_1^2 + 4x_1 + 4 + y_1^2 - 6y_1 + 9) = (2x_1 + 3y_1 - 4)^2$$

$$9x_1^2 - 12x_1y_1 + 4y_1^2 + 68x_1 - 54y_1 + 153 = 0$$

Locus of $P(x_1, y_1)$ is

$9x^2 - 12xy + 4y^2 + 68x - 54y + 153 = 0$

Length of the latus rectum = 4a

2a =Perpendicular distance from S on directrix = $\frac{|2(-2) + 3\cdot 3 - 4|}{\sqrt{4+9}} = \frac{1}{\sqrt{13}}$

Length of the latus rectum = $4a = \frac{2}{\sqrt{3}}$

The axis is perpendicular to the directrix

Equation of the directrix can be taken as

$3x - 2y + k = 0$

This line passes through S(–2, 3)

$-6 - 6 + k = 0 \Rightarrow k = 12$

Equation of the axis is : $3x - 2y + 12 = 0$

2. Prove that the area of the triangle inscribed in the parabola $y^2 = 4ax$ is $\frac{1}{8a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$ sq.units where y_1, y_2, y_3 are the ordinates of its vertices.

Sol.

Given parabola is $y^2 = 4ax$

let $P(at_1^2, 2at_1),\ Q(at_2^2, 2at_2),\ R(at_3^2, 2at_3)$ be the vertices of ΔPQR.

Area of $\Delta PQR = = \frac{1}{2}\begin{vmatrix} at_1^2 - at_2^2 & at_2^2 - at_3^2 \\ 2at_1 - 2at_2 & 2at_2 - 2at_3 \end{vmatrix} = \frac{1}{2}\left|2a^2\left(t_1^2 - t_2^2\right)(t_2 - t_3) - 2a^2\left(t_2^2 - t_3^2\right)(t_1 - t_2)\right|$

$= a^2\left|(t_1 - t_2)(t_2 - t_3)\left(t_1 + t_2 - t_2 - t_3\right)\right|$

$= a^2\left|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right|$

$= \frac{a^3}{a}\left|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right|$

$= \frac{1}{8a}|(2at_1 - 2at_2)(2at_2 - 2at_3)(2at_3 - 2at_1)|$

$= \frac{1}{8a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$

Where $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3)$ are the vertices of ΔPQR.

3. Find the coordinates of the vertex and focus, equation of the directrix and axis of the following parabolas. i) $y^2 + 4x + 4y - 3 = 0$ ii) $x^2 - 2x + 4y - 3 = 0$

Sol. i) given parábola is $y^2 + 4x + 4y - 3 = 0$

$\Rightarrow y^2 + 4y = -4x + 3$

$\Rightarrow (y + 2)^2 - 4 = -4x + 3$

$\Rightarrow (y + 2)^2 = -4x + 7$

$\Rightarrow [y - (-2)]^2 = -4\left[x - \frac{7}{4}\right]$

$h = \frac{7}{4},\ k = -2,\ a = 1$

Vertex $A(h, k) = \left(\frac{7}{4}, -2\right)$

Focus (h–a, k) = $\left(\frac{7}{4}-1,-2\right)=\left(\frac{3}{4},-2\right)$

Equation of the directrix : x – h – a = 0

$x-\frac{7}{4}-1=0 \Rightarrow 4x-11=0$

Equation of the axis is : y – k = 0 ⇒ y + 2 = 0

ii) Given parábola is $x^2 - 2x + 4y - 3 = 0$

⇒ $x^2 - 2x = -4y + 3$

⇒ $(x - 1)^2 - 1 = -4y + 3$

⇒ $(x - 1)^2 = -4y + 4$

⇒ $(x - 1)^2 = -4[y - 1]$

h = 1, k = 1, a = 1

Vertex A(h, k) = (1, 1)

Focus (h, k – a) = (1, 1 – 1) = (1, 0)

Equation of the directrix : y – k – a = 0

y – 1 – 1 = 0 ⇒ y – 2 = 0

Equation of the axis is, x – h = 0 ⇒ x – 1 = 0.

THEOREM

The equation of the chord joining the two points $A(x_1, y_1)$, $B(x_2, y_2)$ on the parabola S = 0 is $S_1 + S_2 = S_{12}$.

THEOREM

The equation of the tangent to the parabola S = 0 at $P(x_1, y_1)$ is $S_1 = 0$.

NORMAL THEOREM

The equation of the normal to the parabola $y^2 = 4ax$ at $P(x_1, y_1)$ is $y_1(x - x_1) + 2a(y - y_1) = 0$.

The equation of the normal to S = 0 at P is : $y_1(x - x_1) + 2a(y - y_1) = 0$

THEOREM

The condition that the line y = mx + c may be a tangent to the parabola $y^2 = 4ax$ is c = a/m.

Proof :

Equation of the parabola is $y^2 = 4ax$ -----(1)

Equation of the line is y = mx + c …(2)

Solving (1) and (2),

$(mx + c)^2 = 4ax \Rightarrow m^2x^2 + c^2 + 2mcx = 4ax$

$\Rightarrow m^2x^2 + 2(mc - 2a)x + c^2 = 0$ which is a quadratic equation in x. therefore it has two roots.

If (2) is a tangent to the parabola, then the roots of the above equation are equal.

$\Rightarrow$ its discreminent is zero

$\Rightarrow 4(mc - 2a)^2 - 4m^2c^2 = 0$

$\Rightarrow m^2c^2 + 4a^2 - 4amc - m^2c^2 = 0$

$\Rightarrow a^2 - amc = 0$

$\Rightarrow a = mc$

$\Rightarrow c = \frac{a}{m}$

II METHOD

Given parabola is $y^2 = 4ax$.

Equation of the tangent is y = mx + c-----(1)

Let $P(x_1, y_1)$ be the point of contact.

The equation of the tangent at P is

$yy_1 - 2a(x + x_1) = 0 \Rightarrow yy_1 = 2ax + 2ax_1$ …(2)

Now (1) and (2) represent the same line.

$\therefore \frac{y_1}{1} = \frac{2a}{m} = \frac{2ax_1}{c} \Rightarrow x_1 = \frac{c}{m}, y_1 = \frac{2a}{m}$

P lies on the line y = mx + c $\Rightarrow y_1 = mx_1 + c$

$\Rightarrow \frac{2a}{m} = m\left(\frac{c}{m}\right) + c \Rightarrow \frac{2a}{m} = 2c \Rightarrow c = \frac{a}{m}$

Note : The equation of a tangent to the parabola
$y^2 = 4ax$ can be taken as y = mx + a/m. and the point of contact is $(a/m^2, 2a/m)$.

COROLLARY

The condition that the line lx + my + n = 0 to touché the parabola $y^2 = 4ax$ is $am^2 = ln$.

Proof :

Equation of the parabola is $y^2 = 4ax$ -----(1)

Equation of the line is lx + my + n = 0

$\Rightarrow y = -\frac{l}{m}x - \frac{n}{m}$

But this line is a tangent to the parabola, therefore

C = a/m $\Rightarrow -\frac{n}{m} = \frac{a}{-l/m} \Rightarrow \frac{n}{m} = \frac{am}{l} \Rightarrow am^2 = ln$

Hence the condition that the line lx + my + n = 0 to touché the parabola $y^2 = 4ax$ is $am^2 = ln$.

Note : The point of contact of $lx + my + n = 0$ with $y^2 = 4ax$ is $(n/l, -2am/l)$.

COROLLARY

The condition that the line $lx + my + n = 0$ to touch the parabola $x^2 = 4ay$ is $al^2 = mn$.

THEOREM

Two tangents can be drawn to a parabola from an external point.

Note

1. If m_1, m_2 are the slopes of the tangents through P, then m_1, m_2 become the roots of equation (1). Hence $m_1 + m_2 = y_1/x_1$, $m_1m_2 = a/x_1$.

2 : If P is a point on the parabola S = 0 then the roots of equation (1) coincide and hence only one tangent can be drawn to the parabola through P.

3 : If P is an internal point to the parabola
S = 0 then the roots of (1) are imaginary and hence no tangent can be drawn to the parabola through P.

THEOREM

The equation in the chord of contact of $P(x_1, y_1)$ with respect to the parabola S = 0 is $S_1 = 0$.

PARAMETRIC EQUATIONS OF THE PARABOLA

A point (x, y) on the parabola $y^2 = 4ax$ can be represented as $x = at^2$, $y = 2at$ in a single parameter t. Theses equations are called parametric equations of the parabola $y^2 = 4ax$. The point $(at^2, 2at)$ is simply denoted by t.

THEOREM

The equation of the tangent at $(at^2, 2at)$ to the parabola is $y^2 = 4ax$ is $yt = x + at^2$.

Proof:

Equation of the parabola is $y^2 = 4ax$.

Equation of the tangent at $(at^2, 2at)$ is $S_1 = 0$.

$\Rightarrow (2at)y - 2a(x + at^2) = 0$

$\Rightarrow 2aty = 2a(x + at^2) \Rightarrow yt = x + at^2$.

THEOREM

The equation of the normal to the parabola $y^2 = 4ax$ at the point t is $y + xt = 2at + at^3$.

Proof :

Equation of the parabola is $y^2 = 4ax$.

The equation of the tangent at t is :

$yt = x + at^2 \Rightarrow x - yt + at^2 = 0$

The equation of the normal at $(at^2, 2at)$ is

$t(x - at^2) + 1(y - 2at) = 0$

$\Rightarrow xt - at^3 + y - 2at = 0 \Rightarrow y + xt = 2at + at^3$

THEOREM

The equation of the chord joining the points t_1 and t_2 on the parabola $y^2 = 4ax$ is $y(t_1+t_2) = 2x + 2at_1t_2$.

Proof :

Equation of the parabola is $y^2 = 4ax$.

Given points on the parabola are

$P(at_1^2, 2at_1),\ Q(at_2^2, 2at_2)$.

Slope of $\overleftrightarrow{PQ}$ is

$$\frac{2at_2 - 2at_1}{at_2^2 - at_1^2} = \frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)} = \frac{2}{t_1 + t_2}$$

The equation of $\overleftrightarrow{PQ}$ is $y - 2at_1 = \dfrac{2}{t_1 + t_2}(x - at_1^2)$.

$\Rightarrow (y - 2at_1)(t_1 + t_2) = 2(x - at_1^2)$

$\Rightarrow y(t_1 + t_2) - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2$

$\Rightarrow y(t_1 + t_2) = 2x + 2at_1t_2$.

Note:

If the chord joining the points t_1 and t_2 on the parabola $y^2 = 4ax$ is a focal chord then $t_1t_2 = -1$.

Proof :

Equation of the parabola is $y^2 = 4ax$

Focus S = (a, o)

The equation of the chord is $y(t_1 + t_2) = 2x + 2at_1t_2$

If this is a focal chord then it passes through the focus (a, 0).

$\therefore\ 0 = 2a + 2at_1t_2 \Rightarrow t_1t_2 = -1$.

THEOREM

The point of intersection of the tangents to the parabola $y^2 = 4ax$ at the points t_1 and t_2 is $(at_1t_2, a[t_1 + t_2])$.

Proof :

Equation of the parabola is $y^2 = 4ax$

The equation of the tangent at t_1 is $yt_1 = x + at_1^2$ …(1)

The equation of the tangent at t_2 is $yt_2 = x + at_2^2$ …(2)

(1) – (2) $\Rightarrow y(t_1 - t_2) = a\left(t_1^2 - t_2^2\right) \Rightarrow y = a(t_1 + t_2)$

(1) $\Rightarrow a(t_1 + t_2)t_1 = x + at_1^2$

$\Rightarrow at_1^2 + at_1t_2 = x + at_1^2 \Rightarrow x = at_1t_2$

$\therefore$ Point of intersection = $(at_1t_2, a[t_1 + t_2])$.

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THEOREM

Three normals can be drawn form a point (x_1, y_1) to the parabola $y^2 = 4ax$.