Integrals

- Integration is the inverse process of differentiation. In the differential calculus, we are given a function and we have to find the derivative or differential of this function, but in the integral calculus, we are to find a function whose differential is given. Thus, integration is a process which is the inverse of differentiation.
- Let $(d/dx)*F(x) = f(x)$. Then we write $\int f(x)\,dx = F(x) + C$. These integrals are called indefinite integrals or general integrals; C is called constant of integration. All these integrals differ by a constant.

- From the geometric point of view, an indefinite integral is collection of family of curves, each of which is obtained by translating one of the curves parallel to itself upwards or downwards along the y-axis.
- Some properties of indefinite integrals are as follows: 1. $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$ 2. For any real number k, $\int k\, f(x)\, dx = k \int f(x)\, dx$
- More generally, if $f_1, f_2, f_3 \ldots f_n$ are functions and $k_1, k_2 \ldots k_n$ are real numbers. Then $\int [k_1 f_1(x) + k_2 f_2(x) + \ldots + k_n f_n(x)]\, dx = k_1 \int f_1(x)\, dx + k_2 \int f_2(x)\, dx + \ldots + k_n \int f_n(x)\, dx$.
- $\int \cos x\, dx = \sin x + C$
- $\int \sin x\, dx = -\cos x + C$
- $\int \sec2\, x\, dx = \tan x + C$

- $\int \text{cosec}2\ x\ dx = -\cot x + C$
- $\int \sec x \tan x\ dx = \sec x + C$
- $\int \text{cosec}\ x \cot x\ dx = -\text{cosec}\ x + C$
- $\int dx/(\sqrt{1} - x^2) = \sin^{-1}x + C$
- $\int dx/(\sqrt{1} - x^2) = -\cos^{-1}x + C$
- $\int dx/(1 + x^2) = \tan^{-1}x + C$
- $\int dx/(1 + x^2) = -\cot^{-1}x + C$
- $\int e^x dx = e^x + C$
- $\int a^x dx = (a^x/\log a)+c$
- $\int dx/(x\sqrt{x^2} - 1) = \sec^{-1}x + C$
- $\int dx/(x\sqrt{x^2} - 1) = -\text{cosec}^{-1}x + C$
- $\int (1/x)dx = \log |x| + C$

- Integration by partial fractions
    - $(px + q)/(x-a)(x-b) = A/(x-a) + B/(x-b)$
    - $(px + q)/(x-a)^2 = A/(x-a) + B/(x-a)^2$
- Integration by substitution
    - $\int \tan x \, dx = \log \sec x + C$
    - $\int \cot x \, dx = \log \sin x + C$
    - $\int \sec x \, dx = \log \sec x + \tan x + C$
    - $\int \mathrm{cosec} x \, dx = \log \mathrm{cosec}\, x - \cot x + C$

- Integrals of some special functions
  - $\int dx/(x^2 – a^2) = (1/2a)\log|(x – a)/(x – b)| + C$
  - $\int dx/(a^2 – x^2) = (1/2a)\log|(a - x)/(b - x)| + C$
  - $\int dx/(x^2 + a^2) = (1/a) \tan^{-1}(x/a) + C$
  - $\int dx/ \sqrt{}(x^2 – a^2) = \log |x + \sqrt{} x^2 – a^2| + C$
  - $\int dx/ \sqrt{}(x^2 + a^2) = \log |x + \sqrt{} x^2 + a^2| + C$
  - $\int dx/ \sqrt{}(x^2 - a^2) = \sin^{-1}(x/a) + C$

- Integration by parts:-

  For given functions f_1 and f_2, we have

  $$\int f_1(x).f_2(x)dx = f_1(x) \int f_2(x)dx - \int[(d/dx)f_1(x). \int f_2(x)dx] + C$$

  Integral of the product of two functions = first function × integral of the second function – integral of {differential coefficient of the first function × integral of the second function}. Care must be taken in choosing the first function and the second function. Obviously, we must take that function as the second function whose integral is well known to us.

- $\int e^x[f (x) + f '(x)] dx = \int e^x f (x) dx + C$

Examples

- Find the integral of $\int \sin^3 x \cos^2 x\, dx$

Solution:-

$\int \sin^3 x \cos^2 x\, dx = \int \sin^2 x \cos^2 x (\sin x)\, dx$

$= \int (1 - \cos 2x) \cos 2x (\sin x)\, dx$

Put $t = \cos x$ so that $dt = - \sin x\, dx$

Therefore, $\int \sin^2 x \cos^2 x (\sin x)\, dx = - \int (1 - t^2) t^2\, dt$

$- \int (t^2 - t^4) dt = - [(t^3/3) - (t^5/5)] + C$

$= (-1/3)\cos^3 x + (1/5)\cos^5 x + C$

- Evaluate $\int dx/(x^2 - 6x + 13)$

Solution:-

$x^2 - 6x + 13 = x^2 - 6x + 32 - 32 + 13 = (x - 3)^2 + 4$

Hence $\int dx/(x^2 - 6x + 13) = \int dx/[(x - 3)^2 + 2^2]$

$x - 3 = t$

$dx = dt$

$\int dx/(x^2 - 6x + 13) = \int dx/(t^2 + 2^2) = (1/2)\tan^{-1}(t/2) + C = (1/2)\tan^{-1}((x - 3)/2)$.