CHAPTER 11

DIFFERENTIAL EQUATIONS

TOPICS:

1. Differential Equation,Order And Degree

2. Formation Of Differential Equation.

3.Variable Seperable Method

4.Homogeneous Method

5.Non Homogeneous Method

6.Linear Differential Equation

7.Bernoulli's Differntial Equation

DIFFERENTIAL EQUATIONS

An equation involving one dependent variable, one or more independent variables and the differential coefficients (derivatives) of dependent variable with respect to independent variables is called a differential equation.

ORDER OF A DIFFERENTIAL EQUATION :

The order of the highest derivative involved in an ordinary differential equation is called the order of the differential equation.

DEGREE OF A DIFFERENTIAL EQUATION

The degree of the highest derivative involved in an ordinary differential equation, when the equation has been expressed in the form of a polynomial in the highest derivative by eliminating radicals and fraction powers of the derivatives is called the degree of the differential equation.

EXERCISE --- 11(A)

I

1. **Find the order of the family of the differential equation obtained by eliminating the arbitrary constants b and c from** $xy = ce^{x} + be^{-x} + x^{2}$ **.**

Sol.

Equation of the curve is $xy = ce^{x} + be^{-x} + x^{2}$

Number of arbitrary constants in the given curve is 2.

Therefore, the order of the corresponding differential equation is 2.

2. **Find the order of the differential equation of the family of all circles with their centers at the origin.**

Given family of curves is $x^2 + y^2 = a^2 ---(1)$, a parameter.

Diff (1) w.r.t x, $2x+2y.y_1 = 0$.

Hence required differential equation is $x+y.y_1 = 0$.

Order of the differential equation is 1.

II

1. **Form the differential equation of the following family of curves where parameters are given in brackets.**

i). $y = c(x-c)^2 ; (c)$

$y = c(x-c)^2 ------(1)$

Diff. w.r.t x,

$y_1 = c.2(x-c) ----(2)$

$$\frac{(1)}{(2)} \Rightarrow \frac{y}{y_1} = \frac{x-c}{2}$$

$$\Rightarrow x - c = \frac{2y}{y_1} \quad \text{and} \quad c = x - \frac{2y}{y_1}$$

$$from (1), y = \left(x - \frac{2y}{y_1} \right)\left(\frac{2y}{y_1} \right)^2 \Rightarrow y_1^3 = 4y\left(xy_1 - 2y \right)$$

ii) $xy = ae^x + be^{-x}; (a,b)$

$$xy = ae^x + be^{-x} ---(1)$$

Diff.w.r.t. x,

$$y + x.y_1 = ae^x - be^{-x} ---(2)$$

$diff .w.r.t.$ x,

$$y_1 + y_1 + xy_2 = ae^x + be^{-x} = xy$$

$$\therefore 2y_1 + xy_2 = xy$$

Which is required differential equation.

iii) $y = (a+bx)e^{kx}; (a,b)$

$$y = (a+bx)e^{kx} ---(1)$$

Diff.w.r.t x,

$$\Rightarrow y_1 = k\left(a+bx\right)e^{kx} + be^{kx}$$

$$\Rightarrow y_1 = ky + be^{kx} ---(2)$$

Diff.w.r.t. x,

$$\Rightarrow y_2 = ky_1 + kbe^{kx}$$

$$\Rightarrow y_2 = ky_1 + k\left(y_1 - ky\right)$$

$\Rightarrow y_2 = 2ky_1 - k^2 y$ which is required differential equation.

iv) $y = a\cos(nx+b); (a,b)$

ans: $y_2 + n^2 y = 0$

2. Obtain the differential equation which corresponds to each of the following family of curves.

i) The rectangular hyperbolas which have the coordinates axes as asymptotes.

Sol. Equation of the rectangular hyperbola is $xy=c^2$ where c is arbitrary constant.

Differentiating w.r.t. x

$$x\frac{dy}{dx}+y=0$$

ii) The ellipses with centres at the origin and having coordinate axes as axes.

Sol. Equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Diff. w.r.t.x,

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0\Rightarrow y.y_1=-\frac{b^2}{a^2}x$$

Diff. w.r.t. x,

$$y.y_2+y_1.y_1=-\frac{b^2}{a^2}\Rightarrow y.y_2+2y_1=\frac{y.y_1}{x}$$

$$\Rightarrow x\left(y.y_2+2y_1\right)=y.y_1$$

III.

1. Form the differential equations of the following family of curves where parameters are given in brackets.

i) $y=ae^{3x}+be^{4x};\ (a,b)$

Sol. $y=ae^{3x}+be^{4x}\ ----(1)$

Differentiating w.r.t x

$$y_1=3ae^{3x}+4be^{4x}\ -----(2)$$

Differentiating w.r.t x,

$$y_2=9ae^{3x}+16be^{4x}\ -----(3)$$

Eliminating a,b from above equations,

$$\begin{vmatrix} y & e^{3x} & e^{4x} \\ y_1 & 3e^{3x} & 4e^{4x} \\ y_2 & 9e^{3x} & 16e^{4x} \end{vmatrix}=0\Rightarrow\begin{vmatrix} y & 1 & 1 \\ y_1 & 3 & 4 \\ y_2 & 9 & 16 \end{vmatrix}=0$$

$\Rightarrow y_2-7y_1+12y=0$ **which is the required differential equation.**

ii) y = ax^2 + bx ; (a, b)

Ans: $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$

iii) $ax^2 + by^2 = 1$; (a, b)

Sol.

Given equation is

$ax^2 + by^2 = 1$------(1)

Differentiating w.r.t. x

$\Rightarrow 2ax + 2byy_1 = 0$

$\Rightarrow ax + byy_1 = 0 ------(2)$

Differentiating w.r.t. x

$\Rightarrow a + b(yy_2 + y_1y_1) = 0 \Rightarrow a + b\left(yy_2 + y_1^2\right) = 0$

$\Rightarrow ax + bx\left(yy_2 + y_1^2\right) = 0$ ----(3)

$(3) - (2) \Rightarrow bx\left(yy_2 + y_1^2\right) - byy_1 = 0$

$\Rightarrow x\left(yy_2 + y_1^2\right) - yy_1 = 0$

iv) $xy = ax^2 + \frac{b}{x}$; (a, b)

Ans: $x^2\left(\frac{d^2y}{dx^2}\right) + 2x\left(\frac{dy}{dx}\right) - 2y = 0$

2. Obtain the differential equation which corresponds to each of the following family of curves.

i) The circles which touch the Y-axis at the origin.

Sol. Equation of the given family of circles is

$x^2 + y^2 + 2gx = 0$, g is arbitrary const …(i)

$x^2 + y^2 = -2gx$

Differentiating w.r.t. x

$2x + 2yy_1 = -2g$ …(ii)

Substituting in (i)

$x^2 + y^2 = x(2x + 2yy_1)$ by (ii)

$= 2x^2 + 2xyy_1$

$yy^2 - 2xyy_1 - 2x^2 = 0$

$y^2 - x^2 = 2xy\frac{dy}{dx}$.

ii) The parabolas each of which has a latus rectum 4a and whose axes are parallel to x-axis.

Sol.

Equation of the given family of parabolas is

$(y - k)^2 = 4a(x - h)$-----(i)

where h,k are arbitrary constants

Differentiating w.r.t. x

$2(y - k)y_1 = 4a$

$(y - k)y_1 = 2a$ …(2)

Differentiating w.r.t. x

$(y - k)y_2 + y_1^2 = 0$ …(3)

From (2), $y - k = \frac{2a}{y_1}$

Substituting in (3)

$$\frac{2a}{y_1} \cdot y_2 + y_1^2 = 0 \Rightarrow 2ay_2 + y_1^3 = 0$$

iii) The parabolas having their foci at the origin and axis along the x-axis.

Sol.

Given family of parabolas is y2 = 4a(x + a)-----(i)

Diff. w.r.t.x,

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{1}{2}yy' = a \text{ -----(2)}$$

From (i) and (2),

$$y^2 = 4\frac{1}{2}yy'\left(x + \frac{1}{2}yy'\right)$$

$$y^2 = 2y'x + 4 \cdot \frac{1}{4}y^2y'^2 \Rightarrow y^2 = 2yy'x + y^2y'^2$$

$$y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) = y$$

SOLUTIONS OF DIFFERENTIAL EQUATIONS

1.VARIABLES SEPARABLE

Let the given equation be $\frac{dy}{dx} = f(x, y)$. If f(x, y) is a variables separable function, i.e., f(x, y) = g(x)h(y) then the equation can be written as $\frac{dy}{dx} = g(x)h(y) \Rightarrow \frac{dy}{h(y)} = g(x)dx$. By integrating both sides, we get the solution of $\frac{dy}{dx} = f(x, y)$. This method of finding the solution is known as variables separable.

EXERCISE – 11(B)

1. Find the general solution of $\sqrt{1-x^2}dy + \sqrt{1-y^2}dx = 0$.

Sol. Given D.E is $\sqrt{1-x^2}dy + \sqrt{1-y^2}dx = 0$

$$\sqrt{1-x^2}dy = -\sqrt{1-y^2}dx$$

Integrating both sides

$$\int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}}$$

$\sin^{-1}y = -\sin^{-1}x + c$

Solution is $\sin^{-1}x + \sin^{-1}y = c$, where c is a constant.

2. Find the general solution of $\frac{dy}{dx} = \frac{2y}{x}$.

Sol. $\frac{dy}{dx} = \frac{2y}{x} \Rightarrow \int \frac{dy}{y} = 2\frac{dx}{x}$

Integrating both sides

$$\log c + \log y = 2\log x$$

$$\log cy = \log x^2$$

Solution is $cy = x^2$ where c is a constant.

II. Solve the following differential equations.

1. $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Sol. $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Integrating both sides

$$\Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\Rightarrow \tan^{-1} y = \tan^{-1} x + \tan^{-1} c$$ where c is a constant.

2. $\frac{dy}{dx} = e^{y-x}$

Sol. $\frac{dy}{dx} = \frac{e^y}{e^x} \Rightarrow \frac{dy}{e^y} = \frac{dx}{e^x}$

Integrating both sides $\int e^{-x} dx = \int e^{-y} dy \Rightarrow -e^{-x} = -e^{-y} + c$

$e^{-y} = e^{-x} + c$ where c is a constant.

3. $(e^x + 1)y\ dy + (y + 1)dx = 0$

Sol. $(e^x + 1)y\ dy = -(y + 1)dx$

$$\frac{ydy}{y+1} = -\frac{dx}{e^x+1}$$

Integrating both sides

$$\int\left(1-\frac{1}{y+1}\right)dy = \int -\frac{e^{-x}dx}{e^{-x}+1}$$

$$y - \log(y+1) = \log(e^{-x}+1) + \log c$$

$$\Rightarrow y - \log(y+1) = \log c(e^{-x}+1)$$

$$\Rightarrow y = \log(y+1) + \log c(e^{-x}+1)$$

$$y = \log c(y+1)(e^{-x}+1)$$

Solution is : $e^y = c(y+1)(e^{-x}+1)$.

4. $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Sol. $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y} = \frac{e^x}{e^y} + \frac{x^2}{e^y}$

Integrating both sides

$$\int e^y \cdot dy = \int (e^x + x^2)dx$$

Solution is : $e^y = e^x + \frac{x^3}{3} + c$

5. tany dx + tanx dy = 0

Sol. tan y dx = –tan x dy

$$\frac{dx}{\tan x}=\frac{-dy}{\tan y}\Rightarrow\frac{\cos x}{\sin x}dx=-\frac{\cos y}{\sin y}dy$$

Taking integration

$$\int\frac{\cos x}{\sin x}dx=-\int\frac{\cos y}{\sin y}dy$$

$$\log\sin x=-\log\sin y+\log c$$

$$\log\sin x+\log\sin y=\log c$$

$$\log(\sin x\cdot\sin y)=\log c\Rightarrow\sin x\cdot\sin y=c$$

6. $\sqrt{1+x^2}dx+\sqrt{1+y^2}dy=0$

Sol. $\sqrt{1+x^2}dx=-\sqrt{1+y^2}dy$

Integrating both sides $\int\sqrt{1+x^2}dx=-\int\sqrt{1+y^2}dy$

$$\frac{x}{2}\times\sqrt{1+x^2}+\frac{1}{2}\sinh^{-1}x=$$

$$y\frac{\sqrt{1+y^2}}{2}=\frac{1}{2}\sinh^{-1}x+c$$

$$x\sqrt{1+x^2}+y\sqrt{1+y^2}+\log\left[(x+\sqrt{1+x^2})(y+\sqrt{1+y^2})\right]=c$$

7. $y-x\frac{dy}{dx}=5\left(y^2+\frac{dy}{dx}\right)$

Sol. $y-5y^2=(x+5)\frac{dy}{dx}\Rightarrow\frac{dx}{x+5}=\frac{dy}{y(1-5y)}$

Integrating both sides

$$\int\frac{dx}{x+5}=\int\frac{dy}{y(1-5y)}=\int\left(\frac{1}{y}+\frac{5}{1-5y}\right)dy$$

$$\ln|x+5|=\ln y-\ln|1-5y|+\ln c$$

$$\ln|x+5|=\ln\left|\frac{cy}{1-5y}\right|\Rightarrow x+5=\left(\frac{cy}{1-5y}\right)$$

8. $\frac{dy}{dx}=\frac{xy+y}{xy+x}$

Sol. $\frac{dy}{dx}=\frac{y(x+1)}{x(y+1)}\Rightarrow\frac{y+1}{y}dy=\frac{x+1}{x}dx$

$$\int\left(1+\frac{1}{y}\right)dy=\int\left(1+\frac{1}{x}\right)dx$$

$$y+\log y=x+\log x+\log c$$

$$y-x=\log\left|\frac{cx}{y}\right|$$

III.

1. $\frac{dy}{dx}=\frac{1+y^2}{(1+x^2)xy}$

Sol. $\frac{dy}{dx}=\frac{1+y^2}{(1+x^2)xy}$

$$\Rightarrow\frac{ydy}{1+y^2}=\frac{dx}{x(1+x^2)}$$

$$\frac{2ydy}{1+y^2}=\frac{2xdx}{x^2(1+x^2)}$$

Integrating both sides

$$\int\frac{2ydy}{1+y^2}=\int\left(\frac{1}{x^2}-\frac{1}{1+x^2}\right)2x\ dx$$

$$\log(1+y^2)=\log x^2-\log(1+x^2)+\log c$$

$$\log(1+x^2)+\log(1+y^2)=\log x^2+\log c$$

Solution is : $(1 + x^2)(1 + y^2) = cx^2$

2. $\frac{dy}{dx}+x^2=x^2\cdot e^{3y}$

Sol. $\frac{dy}{dx}+x^2=x^2\cdot e^{3y}$

$$\Rightarrow\frac{dy}{dx}=x^2\cdot e^{3y}-x^2=x^2(e^{3y}-1)$$

Integrating both sides

$$\int\frac{dy}{e^{3y}-1}=\int x^2dx\Rightarrow\int\frac{e^{-3y}}{1-e^{-3y}}=\int x^2dx$$

$$\log\frac{(1-e^{-3y})}{3}=\frac{x^3}{3}+c$$

$$\log(1-e^{-3y})=x^3+c' \ (c'=3c)$$

Solution is: $1-e^{-3y}=e^{x^3}\cdot k \ \ (k=e^{c'})$

3. **$(xy^2 + x)dx + (yx^2 + y)dy = 0$**

Sol. $(xy^2 + x)dx + (yx^2 + y)dy = 0$

$x(y^2 + 1)dx + y(x^2 + 1)dy = 0$

Dividing with $(1 + x^2)(1 + y^2)$

$$\frac{xdx}{1+x^2}+\frac{ydy}{1+y^2}=0$$

Integrating both sides

$$\int\frac{xdx}{1+x^2}+\int\frac{ydy}{1+y^2}=0$$

$$\frac{1}{2}\left[(\log(1+x^2)+\log(1+y^2)\right]=\log c$$

$$\log(1+x^2)(1+y^2)=2\log c=\log c^2$$

$(1 + x^2)(1 + y^2) = k$ when $k = c^2$.

4. $\frac{dy}{dx}=2y\tanh x$

Sol. $\frac{dy}{dx}=2y\tanh x\Rightarrow\frac{dy}{y}=2\tanh x dx$

Integrating both sides

$$\int\frac{dy}{y}=2\int\tanh x \ dx$$

$$\log y=2\log|\cosh x|+\log c$$

$$\ln y=2\ln\cosh x+\ln c\Rightarrow y=c\cos h^2 x$$

5. $\sin^{-1}\left(\frac{dy}{dx}\right)=x+y$

$$\frac{dy}{dx}=\sin(x+y)\Rightarrow x+y=t$$

$$1+\frac{dy}{dx}=\frac{dt}{dx}$$

$$\frac{dt}{dx}-1=\sin t\Rightarrow\frac{dt}{dx}=1+\sin t$$

$$\frac{dt}{1+\sin t} = dx$$

Integrating both sides

$$\int \frac{dt}{1+\sin t} = \int dx$$

$$\int \frac{1-\sin t}{\cos^2 t} dt = x + c$$

$$\int \sec^2 t dt - \int \tan t \cdot \sec t \, dt = x + c$$

$$\tan t - \sec t = x + c$$

$$\Rightarrow \tan(x+y) - \sec(x+y) = x + c$$

6. $\dfrac{dy}{dx} + \dfrac{y^2+y+1}{x^2+x+1} = 0$

$$\frac{-dy}{y^2+y+1} = \frac{dx}{x^2+x+1}$$

Integrating both sides

$$-\int \frac{dy}{y^2+y+1} = \int \frac{dx}{x^2+x+1}$$

$$-\int \frac{dy}{\left(y+\frac{1}{2}\right)^2 + \frac{3}{4}} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$-\frac{2}{\sqrt{3}} \tan^{-1} \frac{(y+1/2)}{\sqrt{3/2}} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{(x+1/2)}{\sqrt{3/2}} + c$$

$$\tan^{-1} \frac{2x+1}{\sqrt{3}} + \tan^{-1} \frac{2y+1}{\sqrt{3}} = c$$

7. $\dfrac{dy}{dx} = \tan^2(x+y)$

Sol. $\dfrac{dy}{dx} = \tan^2(x+y)$ put v = x + y

$$\frac{dv}{dx} = 1 + \frac{dy}{dx} = 1 + \tan^2 v = \sec^2 v$$

$$\int \frac{dv}{\sec^2 v} = \int dx = \int \cos^2 v \cdot dv = x + c$$

$$\int \frac{(1+\cos 2v)}{2} dv = x + c$$

$$\Rightarrow \int (1+\cos 2v) dv = 2x + 2c$$

$$v+\frac{\sin 2v}{2}=2x+2c$$
$$2v+\sin 2v=4x+c'$$
$$2(x+y)+\sin 2(x+y)=4x+c'$$
$$x-y-\frac{1}{2}\sin[2(x+y)]=c$$

Homogeneous Differential Equations :

A differential equation $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ is said to be a homogeneous differential equation in x, y if both f(x, y), g(x, y) are homogeneous functions of same degree in x and y.

To find the solution of the h.d.e put y = vx, then $\frac{dy}{dx}=v+x\frac{dv}{dx}$. Substituting these values in given differential equation, then it reduces to variable separable form. Then we find the solution of the D.E.

NOTE: Some times to solve the give homogeneous differential equation, we take the substitution x =vy.

EXERCISE – 11(C)

1. Express $xdy-ydx=\sqrt{x^2+y^2}dx$ **in the form** $F\left(\frac{y}{x}\right)=\frac{dy}{dx}$**.**

Sol. $x\cdot dy-ydx=\sqrt{x^2+y^2}dx$

$$x\frac{dy}{dx}-y=\sqrt{x^2+y^2}\Rightarrow x\frac{dy}{dx}=y+\sqrt{x^2+y^2}$$

$$\frac{dy}{dx}=\frac{y}{x}+\sqrt{\frac{x^2+y^2}{x^2}}=\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}$$

Which is of the form $F\left(\frac{y}{x}\right)=\frac{dy}{dx}$

2. Express $\left(x-y\mathrm{Tan}^{-1}\frac{y}{x}\right)dx+x\tan^{-1}\frac{y}{x}dy=0$ **in the form** $F\left(\frac{y}{x}\right)=\frac{dy}{dx}$**.**

Ans: $\frac{dy}{dx}=\frac{\frac{y}{x}\cdot\tan^{-1}\left(\frac{y}{x}\right)-1}{\tan^{-1}\left(\frac{y}{x}\right)}=F\left(\frac{y}{x}\right)$

3. **Express** $x \cdot \frac{dy}{dx} = y(\log y - \log x + 1)$ **in the form** $F\left(\frac{y}{x}\right) = \frac{dy}{dx}$ **.**

Ans: $\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x} + 1\right)$

II. Solve the following differential equations.

1. $\frac{dy}{dx} = \frac{x-y}{x+y}$

Sol. $\frac{dy}{dx} = \frac{x-y}{x+y}$ -----(1)

(1) is a homogeneous D.E.

Put y = vx

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x - vx}{x + vx} = \frac{x(1-v)}{x(1+v)}$$

$$x \cdot \frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-v-v-v^2}{1+v} = \frac{1-2v-v^2}{1+v}$$

$$\int \frac{(1+v)dv}{1-2v-v^2} = \int \frac{dx}{x}$$

$$-\frac{1}{2}\log(1-2v-v^2) = \log x + \log c$$

$$-\frac{1}{2}\log\left(1 - 2 \cdot \frac{y}{x} - \frac{y^2}{x^2}\right) = \log cx$$

$$\log\frac{(x^2 - 2xy - y^2)}{x^2} = -2\log cx = \log(cx)^{-2}$$

$$\frac{x^2 - 2xy - y^2}{x^2} = (cx)^{-2} = \frac{1}{c^2x^2}$$

$$(x^2 - 2xy - y^2) = \frac{1}{c^2} = k$$

2. $\mathbf{(x^2 + y^2)dy = 2xy\ dx}$

Sol. $\frac{dy}{dx} = \frac{2xy}{x^2+y^2}$ which is a homogeneous D.E.

Put y = vx

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

$$v + x \cdot \frac{dv}{dx} = \frac{2x(vx)}{x^2+v^2x^2} = \frac{2v}{1+v^2}$$

$$x \cdot \frac{dv}{dx} = \frac{2v}{1+v^2} - v = \frac{2v - v - v^3}{1+v^2} = \frac{v - v^3}{1+v^2}$$

$$\int \frac{1+v^2}{v(1-v^2)} dv = \int \frac{dx}{x}$$

Let $\frac{1+v^2}{v(1-v^2)} = \frac{A}{v} + \frac{B}{1+v} + \frac{C}{1-v}$

$1+v^2 = A(1-v^2) + BV(1-v) + CV(1+v)$

$v = 0 \Rightarrow 1 = A$

$v = 1 \Rightarrow 1+1 = C(2) \Rightarrow c = 1$

$v = -1 \Rightarrow 1+1 = B(-1)(2) \Rightarrow 2 = -2B \Rightarrow B = -1$

$$\int \frac{1+v^2}{v(1-v^2)} dv = \int \frac{dv}{v} - \int \frac{dv}{1+v} + \int \frac{dv}{1-v}$$

$$= \log v - \log(1+v) - \log(1-v) = \log \frac{v}{1-v^2}$$

$$\therefore \log \frac{v}{1-v^2} = \log x + \log c = \log cx$$

$$\frac{v}{1-v^2} = cx \Rightarrow v = cx(1-v^2)$$

$$\frac{y}{x} = cx\left(1 - \frac{y^2}{x^2}\right) \Rightarrow \frac{y}{x} = cx\frac{(x^2-y^2)}{x^2}$$

Solution is: $y = c(x^2 - y^2)$

3. $\frac{dy}{dx}=\frac{-(x^2+3y^2)}{(3x^2+y^2)}$

Sol. $\frac{dy}{dx}=\frac{-(x^2+3y^2)}{(3x^2+y^2)}$ which is a homogeneous D.E.

Put y = vx

$$\frac{dy}{dx}=v+x\cdot\frac{dv}{dx}$$

$$v+x\cdot\frac{dv}{dx}=\frac{-(x^2+3v^2x^2)}{3x^2+v^2x^2}=\frac{-x^2(1+3v^2)}{x^2(3+v^2)}$$

$$x\cdot\frac{dv}{dx}=-v-\frac{1+3v^2}{3+v^2}$$

$$=\frac{-3v-v^3-1-3v^2}{3+v^2}=-\frac{(v+1)^3}{3+v^2}$$

$$\frac{3+v^2}{(v+1)^3}=\frac{-dx}{x}$$

$$\frac{3+v^2}{(v+1)^3}=\frac{A}{v+1}+\frac{B}{(v+1)^2}+\frac{C}{(v+1)^3}$$

Multiplying with $(v + 1)^3$

$3 + v^2 = A(v + 1)^2 + B(v + 1) + C$

$v = -1 \Rightarrow 3 + 1 = C \Rightarrow C = 4$

Equating the coefficients of v^2

A = 1

Equating the coefficients of V

0 = 2A + B

B = –2A = –2

$$\frac{v^2+3}{(v+1)^3}=\frac{1}{v+1}-\frac{2}{(v+1)^2}+\frac{4}{(v+1)^3}$$

$$\int\frac{v^2+3}{(v+1)^3}=-\int\frac{dx}{x}$$

$$\int\left(\frac{1}{v+1}-\frac{2}{(v+1)^2}+\frac{4}{(v+1)^3}\right)dv=-\log x+\log c\ \log(v+1)+\frac{2}{v+1}-\frac{4}{2(v+1)^2}=\log\frac{c}{x}$$

Solution is:

$$\log\left(\frac{y}{x}+1\right)+\frac{2}{\frac{y}{x}+1}-\frac{2}{\left(\frac{y}{x}+1\right)^2}=\log\frac{c}{x}$$

$$\frac{2x}{x+y}-\frac{2x^2}{(x+y)^2}=\log\frac{c}{x}-\log\frac{(x+y)}{x}$$

$$\frac{2x^2+2xy-2x^2}{(x+y)^2}=\log\frac{c}{x+y}$$

$$\frac{2xy}{(x+y)^2}=\log\frac{c}{x+y}$$

$$\log\left(\frac{x+y}{c}\right)c=-\log\left(\frac{c}{x+y}\right)=-\frac{2xy}{(x+y)^2}$$

4. $y^2\,dx+(x^2-xy)dy=0$

Sol. $y^2\,dx+(x^2-xy)dy=(xy-x^2)dy$

$$\frac{dy}{dx}=\frac{y^2}{xy-x^2}$$ which is a homogeneous D.E.

Let $y=vx\Rightarrow\frac{dy}{dx}=v+x\cdot\frac{dv}{dx}$

$$v+x\frac{dv}{dx}=\frac{v^2x^2}{x^2(v-v^2)}$$

$$x\cdot\frac{dv}{dx}=\frac{v^2}{v-1}-v=\frac{v^2-v^2+v}{v-1}$$

$$\frac{v-1}{v}dv=\frac{dx}{x}\Rightarrow\int\left(1-\frac{1}{v}\right)dv=\int\frac{dx}{x}$$

$$v-\log v=\log x+\log k$$

$$v=\log v+\log x+\log k=\log k(vx)$$

$$\frac{y}{x}=\log ky\Rightarrow ky=e^{y/x}$$

5. $$\frac{dy}{dx}=\frac{(x+y)^2}{2x^2}$$

Sol. $\frac{dy}{dx}=\frac{(x+y)^2}{2x^2}$ which is a homogeneous D.E.

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{(x+vx)^2}{2x^2} = x^2\frac{(1+v)^2}{2x^2}$$

$$x\frac{dv}{dx} = \frac{(1+v^2)}{2} - v = \frac{1+v^2+2v-2v}{2}$$

$$2\int\frac{dv}{1+v^2} = \int\frac{dx}{x} \Rightarrow 2\tan^{-1}v = \log x + \log c$$

$$2\tan^{-1}\left(\frac{y}{x}\right) = \log cx$$

6. $(x^2 - y^2)dx - xy\,dy = 0$

Ans: $x^2(x^2 - 2y^2) = k$

7. $(x^2y - 2xy^2)dx = (x^3 - 3x^2y)dy$

Sol. $(x^2y - 2xy^2)dx = (x^3 - 3x^2y)dy$

$$\frac{dy}{dx} = \frac{x^2y-2xy^2}{x^3-3x^2y}$$ which is a homogeneous D.E.

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^3v-2v^2x^3}{x^3-3vx^3}$$

$$= \frac{(v-2v^2)x^3}{(1-3v)x^3} = \frac{v-2v^2}{1-3v}$$

$$x\frac{dv}{dx} = \frac{v-2v^2}{1-3v} - v$$

$$= \frac{v-2v^2-v(1-3v)}{1-3v} = \frac{-2v^2+3v^2}{1-3v}$$

$$x\frac{dv}{dx} = \frac{v^2}{1-3v} \Rightarrow \frac{1-3v}{v^2}dv = \frac{dx}{x}$$

$$\int\left(\frac{1}{v^2} - \frac{3}{v}\right)dv = \int\frac{dx}{x}$$

$$\frac{-1}{v} - 3\log v = \log x + \log c$$

$$\frac{-x}{y} = 3\log\left(\frac{y}{x}\right) = \log x + \log c$$

$$\frac{-x}{y} - \log\left(\frac{y}{x}\right)^3 = \log xc$$

$$\frac{-x}{y} = \log xc + \log\frac{y^3}{x^3}$$

$$\frac{-x}{y} = \log\left(cx \cdot \frac{y^3}{x^3}\right) = \log\left(\frac{cy^3}{x^2}\right)$$

$$\frac{cy^3}{x^2} = e^{-x/y} \Rightarrow cy^3 = \frac{x^2}{e^{x/y}}$$

$$cy^3 \cdot e^{x/y} = x^2$$

8. $\mathbf{y^2\,dx + (x^2 - xy + y^2)dy = 0}$

Ans: $y = c \cdot e^{\tan^{-1}(y/x)}$

9. $\mathbf{(y^2 - 2xy)dx + (2xy - x^2)dy = 0}$

Ans: $xy(y - x) = c$

10. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

Sol. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$ which is a homogeneous D.E.

Put y = vx $\Rightarrow$ $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} + v = \frac{v^2x^2}{x^2} \Rightarrow x\frac{dv}{dx} = v^2 - 2v$$

$$\frac{dv}{v^2 - 2v} = \frac{dy}{x}$$

Let $\frac{1}{v^2 - 2v} = \frac{A}{v} + \frac{B}{v-2}$

$$1 = A(v-2) + Bv$$

$$v = 0 \Rightarrow 1 = A(-2) \Rightarrow -\frac{1}{2}$$

$$v = 2 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$$

$$-\frac{1}{2}\int\left(\frac{1}{v} - \frac{1}{v-2}\right)dv = \int\frac{dx}{x}$$

$$-\frac{1}{2}\left[\log v - \log(v-2)\right] = \log x + \log c$$

$$-\frac{1}{2}\left[\log\frac{v}{v-2}\right] = \log cx$$

$$\log\frac{v}{v-2} = -\log cx = \log(cx)^{-2}$$

$$\frac{v}{v-2} = (cx)^{-2} \Rightarrow \frac{(y/x)}{(y/x)-2} = \frac{1}{c^2x^2}$$

$$\frac{y}{y-2x} = \frac{1}{c^2x^2} \Rightarrow x^2y = \frac{1}{c^2}(y-2x)$$

Solution is :

$y - 2x = c^2x^2y = kx^2y$ where k = c^2

11. $xdy - ydx = \sqrt{x^2+y^2}\,dx$

Ans: $y + \sqrt{x^2+y^2} = cx^2$

12. (2x – y)dy = (2y – x)dx

Ans; (y – x) = c^2 $(x + y)^3$.

13. $(x^2 - y^2)\frac{dy}{dx} = xy$

Ans: x^2 + $2y^2$ (c + log y) = 0.

14. Solve $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

Ans: (y – x) = cx^2y

III.

1. Solve $\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$**.**

Sol. $\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$

$$\Rightarrow \frac{dx}{dy}=-\frac{e^{x/y}\left(1-\frac{x}{y}\right)}{\left(1+e^{x/y}\right)} \text{ which is a homogeneous D.E.}$$

Put $\quad x = vy \Rightarrow \frac{dx}{dy}=v+y\frac{dv}{dy}$

$$(1+e^{v})\frac{dx}{dy}+e^{v}(1-v)=0$$

$$(1+e^{v})\left(v+y\frac{dv}{dy}\right)+e^{v}(1-v)=0$$

$$v+ve^{v}+y(1+e^{v})\frac{dv}{dy}+e^{v}-ve^{v}=0$$

$$y(1+e^{v})dv=-(v+e^{v})dy$$

$$\int\frac{1+e^{v}}{v+e^{v}}dv=-\int\frac{dy}{y}$$

$$\log(v+e^{v})=-\log y+\log c\Rightarrow v+e^{v}=\frac{c}{y}$$

$$\frac{x}{y}+e^{x/y}=\frac{c}{y}\Rightarrow x+y\cdot e^{x/y}=c$$

2. Solve : $x\sin\frac{y}{x}\cdot\frac{dy}{dx}=y\sin\frac{y}{x}-x$

Sol. $x\sin\frac{y}{x}\cdot\frac{dy}{dx}=y\sin\frac{y}{x}-x$

$$\Rightarrow \frac{dy}{dx}=\frac{\frac{y}{x}\left(\sin\left(\frac{y}{x}\right)-\frac{x}{y}\right)}{\sin\left(\frac{y}{x}\right)}$$

Put $y = vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{v\left(\sin v - \frac{1}{v}\right)}{\sin v}$$

$$x\frac{dv}{dx} = \frac{v\sin v - 1 - v\sin v}{\sin v}$$

$$-\sin v dv = \frac{1}{x}dx$$

$$\Rightarrow \int -\sin v \cdot dv = +\int \frac{dx}{x}$$

$$\Rightarrow \cos v = \log x + \log c = \log cx$$

$$\Rightarrow cx = e^{\cos v} = e^{\cos(y/x)}.$$

3. **Solve:** $xdy = \left(y + x\cos^2\frac{y}{x}\right)dx$ **.**

Ans: $\tan\left(\frac{y}{x}\right) = \log x + c$.

4. **Solve : (x – y log y + y log x)dx + x(log y – log x)dy = 0.**

Ans: $= (x - y)\log x + y\log y$

5. **Solve** $(ydx + xdy)x\cos\frac{y}{x} = (xdy - ydx)y\sin\frac{y}{x}$

Ans: $xy\cos\left(\frac{y}{x}\right) = c$

6. **Find the equation of a curve whose gradient is** $\frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x}$ **, where x > 0, y > 0 and which passes through the point (1, π/4).**

Sol. $\frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x}$ which is homogeneous differential equation.

Put y = vx

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v - \cos^2 v \Rightarrow \int\frac{dv}{\cos^2 v} = -\int\frac{dx}{x}$$

$$\int \sec^2 v = -\int\frac{dx}{x} \Rightarrow \tan v = -\log|x| + c$$

This curve passes through (1, π/4)

$$\tan\left(\frac{\pi}{4}\right) = c - \log 1 \Rightarrow c = 1$$

Equation of the curve is:

$$\tan v = 1 - \log|x| \Rightarrow \tan\left(\frac{y}{x}\right) 1 - \log|x|$$

Equations Reducible to Homogeneous Form -Non Homogeneous Differential Equations

The differential equation of the form $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$ is called non homogeneous differential equation.