RAJIV GANDHI UNIVERSITY OF KNOWLEDGE TECHNOLOGIES

SUBJECT: MATHEMATICS | **PUC - SEMESTER: II**

DATE : 06-05-2011 | **MAXIMUM MARKS: 20**

REMEDIAL EXAM

Section: B (Scheme of valuation)

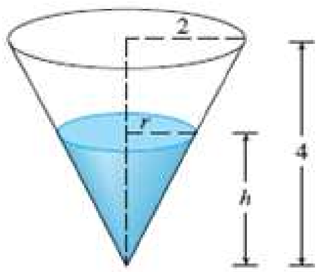
Answer any *two* Questions $(\mathbf{2 \times 5 = 10})$

1.

a. A water tank has the shape of an inverted circular cone with base radius $2m$ and height $4m$. If water is being pumped into the tank at a rate of $2m^3/min$, find the rate at which the water level is rising when the water is $3m$ deep. $2\ Marks$

Solution:

Let V, r and h be the volume of the water, the radius of the surface, and the height of the water at time t, where t is measured in minutes.



Given, $dV/dt = 2m^3/min$

We are asked to find $\frac{dh}{dt}$ when h is $3m$.

The quantities V and h are related by the equation

$$V = \frac{1}{3}\pi r^2 h$$

To express V as a function of h alone.

In order to eliminate r, we use the similar triangles to write

$$\frac{r}{h} = \frac{2}{4} \qquad\qquad r = \frac{h}{2}$$

The expression for V becomes

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3 \qquad\qquad 1Mark$$

Differentiate both sides with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 3\ m$ and $\frac{dV}{dt} = 2m^3/min$,

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2}.2 = \frac{8}{9\pi}m/min$$

The water level is rising at a rate of

$\frac{8}{9\pi}m/min \approx 0.28m/min.$ $\qquad 1Mark$

b. Graph the function $y = x^{2/5}$. $\quad 3\ Marks$

Solution:

Given $y = f(x) = x^{2/5}$.

The domain of f is $(-\infty, \infty)$ and it is continuous on it. Since f is even function of x, its graph is symmetric with respect to the y-axis.

$y' = \frac{2}{5}x^{-3/5}$:

Critical point is $x = 0$ (y' is undefined).

$y'' = \frac{2}{5}\left(\frac{-3}{5}x^{-8/5}\right) = -\frac{6}{25}x^{-8/5}$

Possible inflection point at $x = 0$ (y'' is undefined).

Rise and fall:

For $-\infty < x < 0$, we have $y' = \frac{2}{5}\, x^{-3/5} < 0$.

For $0 < x < \infty$, we have $y'' = \frac{2}{5}\, x^{-3/5} > 0$.

The graph is rises on $(0, \infty)$ and falls on $(-\infty, 0)$.

There is a local minimum at $x = 0$ but there is no local maximum. $\quad 1Mark$

Concavity:

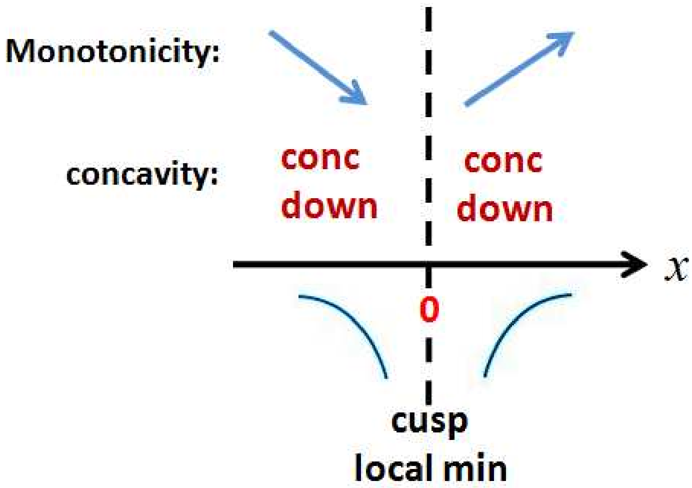
Notice that $y' = -\frac{6}{25x^{8/5}} < 0$ for $x \in \ (-\infty, 0)$ and

$x \ \in \ (0, \infty)$.

The graph is concave down on $(-\infty, 0)$ and $(\ 0, \infty)$.

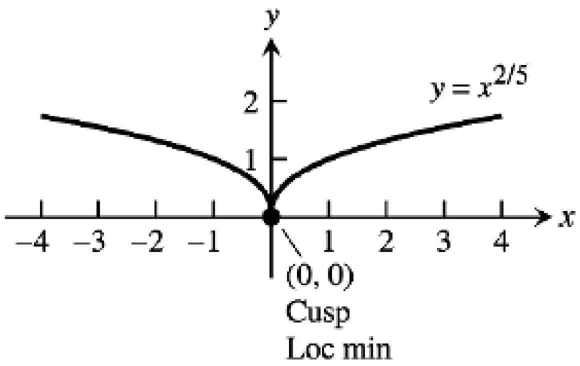
The concavity does not changes at $x = 0$ and $y' \to -\infty$ as $x \to 0^-$, $y' \to \infty$ as $x \to 0^+$ tells us that the graph has a cusp at $x = 0$. $\color{red}{1 Mark}$

Summary:



specific points and sketching the curve

The curve passes through $(0,0)$.



$1 Mark$

2.

a. Find all the asymptotes of the graph of

$$f(x) = \frac{2x^2+x+1}{x+1}. \qquad 2\ Marks$$

Solution:

Notice that $f(x)$ is a rational function and the degree of the numerator is one greater than the degree of the denominator.

We write the rational function (in dominant terms) as a polynomial plus remainder.

$$f(x) = \frac{2x^2+x+1}{x+1} = 2x - 1 + \frac{2}{x+1} \qquad 1 Mark$$

To find the asymptotes, we study the behavior of $f(x)$ as $x \to \pm\infty$ and as $x \to -1$ where the denominator of $f(x)$ is zero.

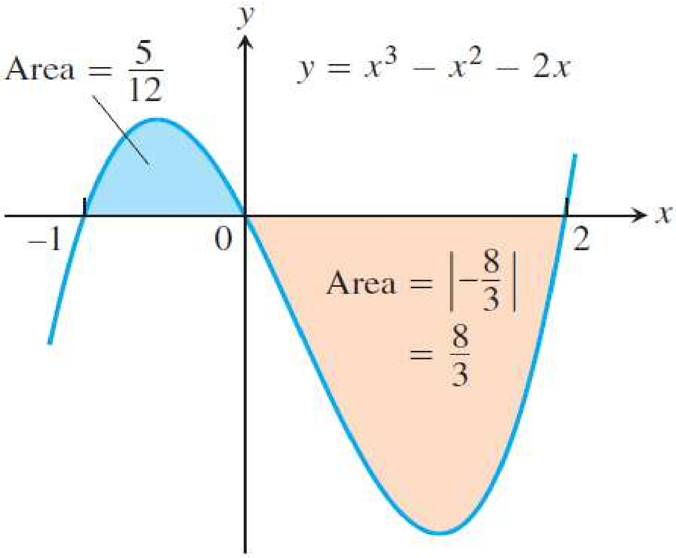
Since $\lim_{x \to -1^+} f(x) = \infty$ and $\lim_{x \to -1^-} f(x) = -\infty$, the line $x = -1$ is a two sided vertical asymptote.

As $x \to \pm\infty$, the remainder approaches 0 and $f(x) \to 2x - 1$.

Therefore, the line $y = 2x - 1$ is an oblique asymptote both to the right and to the left. $1 Mark$

b. Find the area of the region between the $x - axis$ and the graph of $f(x) = x^3 - x^2 - 2x$,$-1 \leq x \leq 2$ $3\ Marks$

Solution:



First find the zeros of f. Since

$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x+1)(x-2)$

the zeros are $x = -1$,0 and 2 .the zeros subdivide $[-1,2]$

$1\ Mark$

into two sub intervals :$[-1,0]$,on which $f \geq 0$ and $[0,2]$,on which $f \leq 0$ .we integrate f over each subinterval and add the absolute values of the calculated integrals.

$$A_1 = \int_{-1}^{0}(x^3 - x^2 - 2x)dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

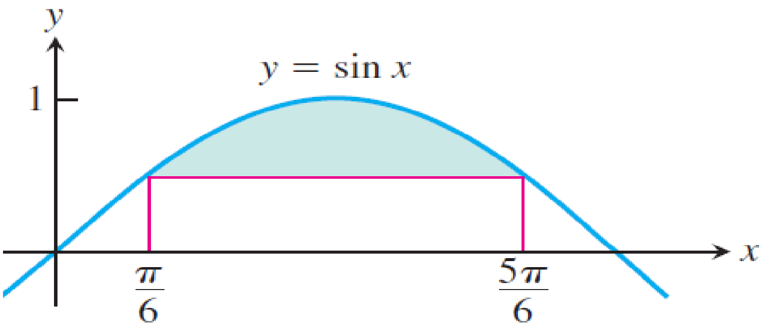
$A_2 = \int_0^2 (x^3 - x^2 - 2x)dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_0^2 = \left[\frac{16}{4} - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$ 1 Mark

The total enclosed area is obtained by adding the absolute values of the calculated integrals

$\therefore A = A_1 + A_2 = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}$ 1 Mark

3.

a. Find the area of the shaded region of the function 2 Marks



Solution :

The area of the rectangle bounded by the lines $x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$

$y = sin\frac{\pi}{6} = \frac{1}{2}$, $y = \sin\frac{5\pi}{6} = \frac{1}{2}$ and $y = 0$ is $\frac{1}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = \frac{\pi}{3}$
$1Mark$

The area under the curve $y = sinx$ on $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ is

$$\int_{\pi/6}^{5\pi/6} \sin x dx = (-\cos x)_{\pi/6}^{5\pi/6} = -\cos\left(\frac{5\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = 2.\frac{\sqrt{3}}{2} = \sqrt{3}$$

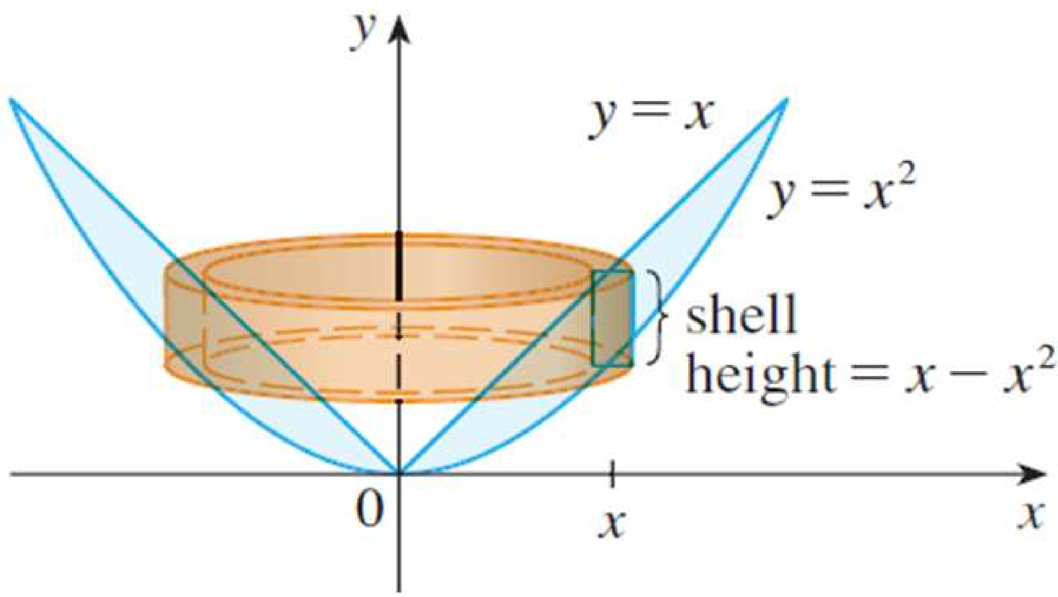
Therefore the area of the shaded region is $\sqrt{3} - \frac{\pi}{3}$. $1\ Mark$

b. Find the volume of the solid generated by revolving the region bounded by the curve $y = x^2$ and the line $y = x$ about the y −axis. 3 Marks

Solution:

Given curve $y = x^2$ and the line is $y = x$.

The region between the curves $y = x^2$ and the line $y = x$ is shown in figure and draw a line segment across it parallel to the axis of revolution: $y -$ axis.



Limits of integration:

$x^2 = x \Longrightarrow x(x-1) = 0 \Longrightarrow x = 0$ and $x = 1$

$\Longrightarrow a = 0$ and $b = 1$ $1\ Mark$

From the figure, $\text{shell radius} = x,\ \text{shell height} = x - x^2$

$$\text{shell thickness} = dx$$

Therefore, the volume of the solid is

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx$$ 1 Mark

$$= \int_0^1 2\pi\, x\left[x - x^2\right]dx$$

$$= 2\pi \int_0^2 \left[x^2 - x^3\right]dx$$

$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left[\left(\frac{1}{3} - \frac{1}{4}\right) - 0\right] = \frac{\pi}{6}$$ 1 Mark

Section-C

Answer any *one* Question $(\mathbf{1} \times \mathbf{10} = \mathbf{10})$

1.

a. State and prove the Mean value Theorem and interpret the conclusion of the theorem geometrically. 5 Marks

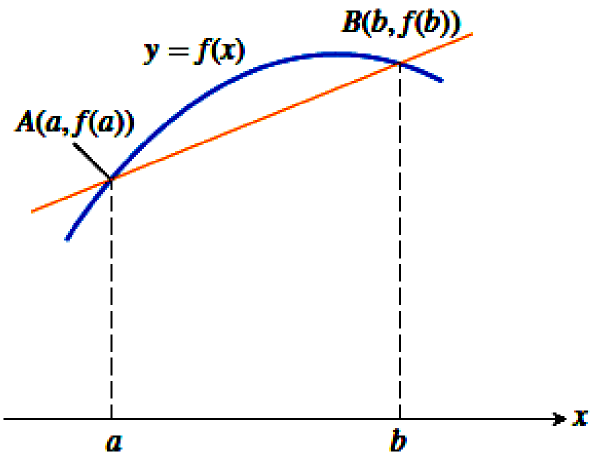
Solution:

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b)-f(a)}{b-a} = f'(c) \text{ --------}(1)$$ 1 Mark

Proof:

We draw the graph of $y = f(x)$ as a curve in the plane and draw a line through the points $A\big(a, f(a)\big)$ and $B\big(b, f(b)\big)$.



The line is the graph of the function $g(x)$, where

$$g(x) - f(a) = \frac{f(b)-f(a)}{b-a}(x-a) \qquad \text{(Point-slope equation)}$$

$$i.e., \quad g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \text{ --------(2)}$$ 1 Mark

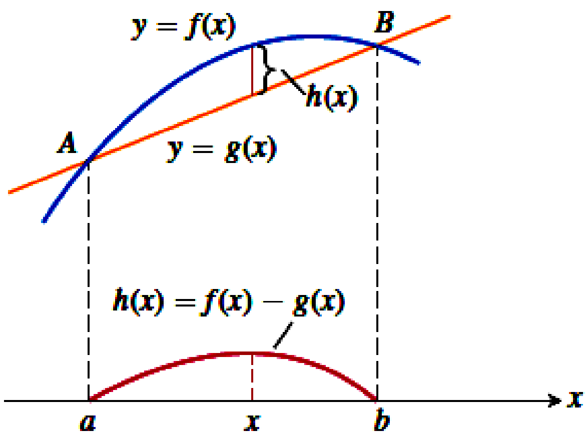
The vertical difference between the graphs of f and g at x is

$$h(x) = f(x) - g(x)$$

$$= f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a) \text{ ----(3)}$$

1 Mark

The figure below shows the graphs of f, g and h together.



The function h is continuous on $[a, b]$ and differentiable on (a, b) because both f and g are. Also $h(a) = h(b) = 0$ because the graphs of f and g both pass through A and B. Thus, the function h satisfies the hypotheses of Rolle's theorem on $[a, b]$.Therefore, $h'(c) = 0$ for some c in (a, b). This is the point we want for equation (1).

We differentiate both sides of equation (3) with respect to x and then set $x = c$.

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

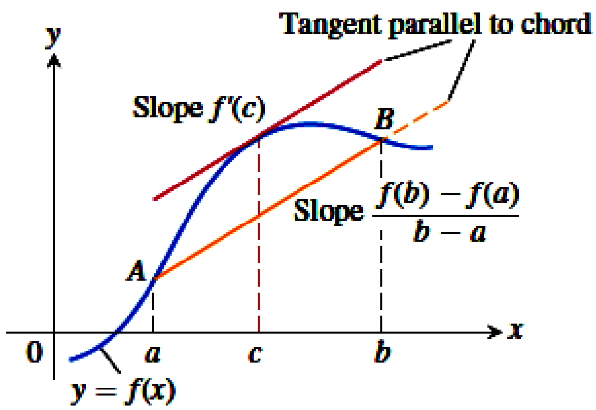
$$0 = f'(c) - \frac{f(b)-f(a)}{b-a}$$

Rearranging,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$ 1 Mark

This proves the theorem.

Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to the chord AB.



1 Mark

b.

Define the Mean value of an integrable function defined on $[a, b]$,State and prove the Mean Value Theorem for definite integrals 5 Marks

Definition

If f is integrable on $[a,b]$, then its ***average (mean) value*** on $[a,b]$ is denoted by $\bar{f}$ or $av(f)$ and

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$$ 1 Mark

Statement:

If f is continuous on $[a,b]$, then at some point c in $[a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$ 1 Mark

Proof

If we divide both sides of the Max-Min inequality by $(b-a)$, we obtain

$$m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$$ 1 Mark

Since f is continuous, the Intermediate Value Theorem for Continuous Functions says that f must assume every value between m and M. 1 Mark

It must therefore assume the value $\frac{1}{b-a}\int_a^b f(x)dx$ at some point c in $[a,b]$.

Hence the theorem . 1 Mark

2.

a. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the curve $x = y - y^3$ and the lines $x = 1,\ y = 1$ about

A. The $x -$ axis

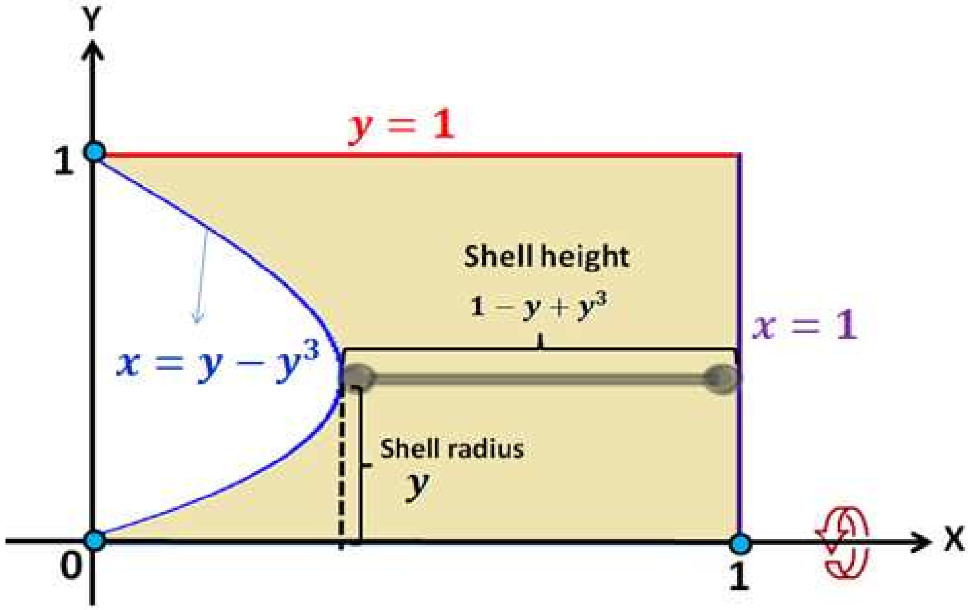
B. The line $x = 1$

C. The line $y = 1$ 5 Marks

Solution:

Given curve $x = y - y^3$ and the lines $x = 1,\ y = 1$.

A. The region between the curve $x = y - y^3$ and the lines $x = 1,\ y = 1$ is as shown in figure and draw a line segment across it parallel to the axis of revolution: x −axis.



Limits of integration:

Since the region in the first quadrant, so $c = 0$ and $d = 1$

From the figure, $\text{shell radius} = y$

$\text{shell height} = 1 - (y - y^3) = 1 - y + y^3,$

$\text{shell thickness} = dy$

Therefore, the volume of the solid is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\, dy$$

$$= \int_0^1 2\pi y\left[1 - y + y^3\right] dy = 2\pi \int_0^1 \left[y - y^2 + y^4\right] dy$$

$$= 2\pi\left[\frac{y^2}{2} - \frac{y^3}{3} + \frac{y^5}{5}\right]_0^1 = 2\pi\left[\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{5}\right) - 0\right] = \frac{11\pi}{15}$$

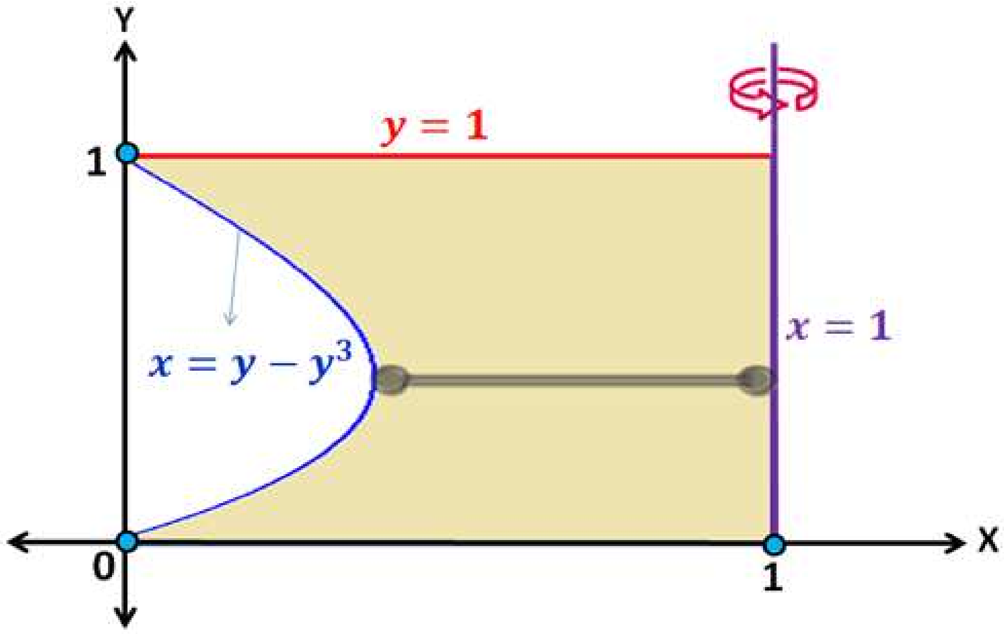
1 Mark

B. In this case, we cannot express y explicitly in terms of x. So, shell method cannot be used. Here we are using the washer method to solve the problem.

The region is shown in figure. Sketch a line segment across it perpendicular to the axis of revolution about the line $x = 1$.

Limits of integration:

Since the region in the first quadrant, so $c = 0$ and $d = 1$



When the region is revolved about y −axis, it will generate a typical washer cross section of the generated solid.

From the figure, inner radius $r(y) = 0$ and outer radius $R(y) = [1 - (y - y^3)]$

Therefore, the cross section's area is

$$A(y) = \pi\left[\left(R(y)\right)^2 - \left(r(y)\right)^2\right]$$

$$= \pi\left[\left(1 - (y - y^3)\right)^2 - 0\right]$$

$$= \pi[1 + y^2 + y^6 - 2y + 2y^3 - 2y^4]$$ 1 Mark

Hence, the volume of the solid is

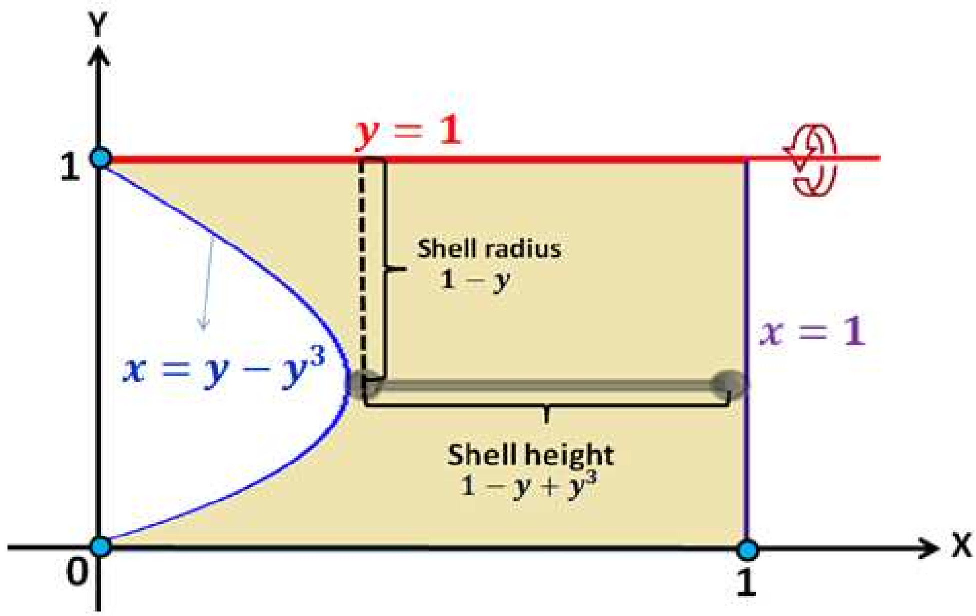
$$V=\int_c^d \pi\left[\left(R(y)\right)^2-\left(r(y)\right)^2\right]dy$$

$$=\int_0^1 \pi\left[1+y^2+y^6-2y+2y^3-2y^4\right]dy$$

$$=\pi\left[y+\frac{y^3}{3}+\frac{y^7}{7}-y^2+\frac{y^4}{2}-\frac{2y^5}{5}\right]_0^1$$

$$=\pi\left[\left(1+\frac{1}{3}+\frac{1}{7}-1+\frac{1}{2}-\frac{2}{5}\right)-0\right]=\frac{121\pi}{210}$$ 1Mark

C. The region is as shown in figure and draw a line segment across it parallel to the axis of revolution: $y=1$.



Limits of integration: $c = 0$ and $d = 1$

From the figure, shell radius $= (1 - y)$

shell height $= 1 - (y - y^3) = 1 - y + y^3$,

shell thickness $= dy$ 1 Mark

Therefore, the volume of the solid is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\, dy$$

$$= \int_0^1 2\pi(1-y)\left[1-y+y^3\right]dy$$

$$= 2\pi\int_0^1 \left[1-y+y^3-y+y^2-y^4\right]dy$$

$$= 2\pi\int_0^1 \left[1-2y+y^2+y^3-y^4\right]dy$$

$$= 2\pi\left[y - y^2 + \frac{y^3}{3} + \frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$$

$$= 2\pi\left[\left(1-1+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}\right)-0\right] = \frac{23\pi}{30}$$

1 Mark

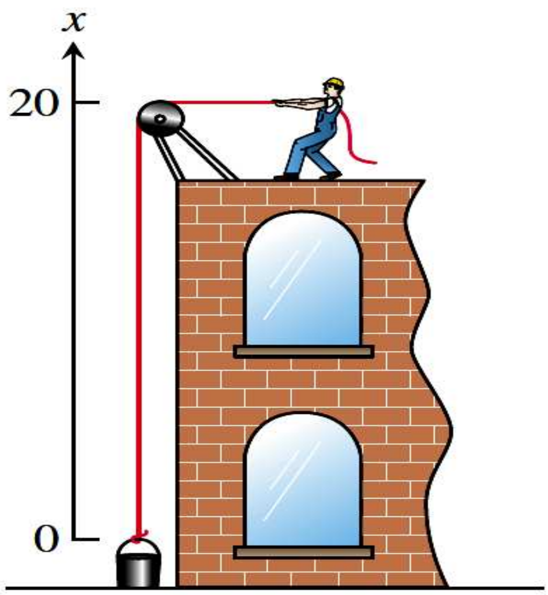
b. A leaky 5-lb bucket is lifted from the ground into the air by pulling in 20 ft of rope at a constant speed .The rope weighs 0.08 lb/ft. The bucket starts with 2 gal of water (16 lb) and leaks at a constant rate. It finishes draining just as it reaches the top. How much work was spent

i) Lifting the water alone;

ii) Lifting the water and bucket together;

iii) Lifting the water, bucket, and rope? 5 Marks

Solution



i) The water alone

The force required to lift the water is equal to the water's weight, which varies steadily from 16 *lb* to 0 *lb* over the 20-*ft* lift. When the bucket is $x\,ft$ off the ground, the water weighs

$$F(x)=16\left(\frac{20-x}{20}\right)=16\left(1-\frac{x}{20}\right)=16-\frac{4x}{5}\ lb$$

1 Mark

The work done is

$$W=\int_a^b F(x)\,dx=\int_0^{20}\left(16-\frac{4x}{5}\right)dx$$

$$=\left[16x-\frac{2x^2}{5}\right]_0^{20}=320-160=160\ ft\text{-}lb$$

1 Mark

ii) The water and bucket together

It takes $5\times20=100$ *ft-lb* to lift a 5-*lb* weight 20 *ft*. Therefore $160+100=260$ *ft-lb* of work was spent lifting the water and bucket together.

1 Mark

iii) The water, bucket, and rope

The total weight of rope at level x is

$$F(x) = (0.08)(20 - x)$$

Work on rope is

$$\int_0^{20} (0.08)(20 - x)\,dx = \int_0^{20} (1.6 - 0.08x)\,dx$$

$$= \left[1.6x - 0.04x^2\right]_0^{20} = 16$$ 1 Mark

The total work for the water, bucket, and rope combined is $160 + 100 + 16 = 276\ ft\text{-}lb$.

1 Mark